

TEACHERS' POINT OF VIEW IN TEACHING MATHEMATICAL PROBLEM-SOLVING

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UDK 37-051:<37.091.33:51>(439)=111

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DOI: 10.15291/magistra.1489

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Received: 28. 8. 2017.

ABSTRACT

There is still a deep gap between the theories of the didactics of mathematics and mathematics teaching practice worldwide. In our article, we analyse our trial to reach practicing mathematics teachers and summarize their opinion about some basic issues of teaching mathematics problem-solving from the point of view of cognitive load theory, what is a quite new topic in mathematics didactics society. We asked on the one hand, teachers from a small town in Hungary, and on the other hand, expert teachers and four young teachers from elite schools in the capital. The four young teachers have also started their PhD studies in mathematics education, besides school teaching. The opinions of the two groups of teachers reflect different attitudes towards teaching problem-solving, but in both cases relevant and important perspectives of the Hungarian school reality. The base of our study was a talk and an article of the first author, related to the role of human memory in learning and teaching mathematical problem-solving. We have been interested in how classroom teachers can take into consideration some results of the cognitive load theory, e.g. the split-attention effect and schema automation in their teaching practice, as well as in their attitudes to the use of worked examples and distributed practice. We analyse the results mostly from the perspective of desirable developments in in-service teacher training in Hungary.

KEY WORDS: *problem solving, working memory, cognitive load, schema automation, worked examples, distributed practise, in-service teacher training, ZDM Classification: B50, C30, D50*

INTRODUCTION

THE HUNGARIAN CONTEXT REGARDING THE MATHEMATICS TEACHERS

There has always been a great difficulty in reaching the practicing teachers, and incorporating their opinions in connection with particular issues into research studies in the field of mathematics didactics. Three trials are analysed.

The survey of teachers' view on the current (2016) state of mathematics education, by the Hungarian Academy of Sciences

The Working Committee on Primary and Secondary Mathematics Education of the Hungarian Academy of Sciences designed and distributed an online questionnaire for practicing mathematics teachers at all levels of K-12 education (grades 1-12) in March 2016, and published the results in May 2016 (Magyar Tudományos Akadémia, 2016). The questions concern specific areas of mathematics education, in precise and detailed forms, strictly connected to the teaching practises of the respondents, also allowing them to express their opinions about other relevant problems of current Hungarian education in general. Approximately 4300 practicing mathematics teachers completed the questionnaire, which is slightly more than one fifth of the number of all practicing mathematics teachers in the country. As we have no place to analyse the answers in details, we summarize only some main points.

The topics of the questions include the mathematics curriculum, the number of mathematics lessons per week at different grades and in different types of schools, the number of teachers' lessons per week, the nationwide exams (at grades 4, 6, 8, and the matura examinations at grade 12), the issue of free choice of mathematics textbooks, fostering mathematically talented pupils (we have two journals, one for grades 3-8 and one for grades 9-12, mathematics circles, weekend and summer camps and competitions), the use of digital technology in mathematics teaching, and the necessity of a nationwide homepage in these topics. Unfortunately, there were no question regarding the lack of a mathematics educational journal for teachers, neither about the absence of a well-organized, obligatory in-service teacher training, and neither about the deficiency of the official control of practising teachers, that is, the lack of professional supervisors in mathematics education. Nobody mentioned the methods of mathematics teaching should be improved.

To summarize the answers, we can formulate that most of the teachers would like (or demand) the mathematics curriculum content—regarding both the range of topics and the depth of some topics—to be decreased, the number of teachers' lessons per week —22–26 lessons by law, for a full-time teacher—to be decreased too, and the number of mathematics lessons for students—3–4 lessons per week—to be increased. The majority of them support the fostering of mathematically talented pupils, that is, organizing the weekend and summer mathematics camps and the mathematics competitions for them. A high proportion of them also mentioned that teachers should be better paid.

National Conference for Mathematics Teachers in Hungary, 2016

Another experience regarding the current Hungarian mathematics educational context was an annual national conference for practicing mathematics teachers, organized in Baja in 2016. About 200 teachers participated on it from the whole country, which is less than one percent of all practicing mathematics teachers. The first author gave a presentation on this conference, including some slides, with the title “Using worked examples in mathematics problem solving teaching.” Unfortunately the presentation was the last one in the conference, so there were only about 50 teachers, mainly experienced, good mathematics teachers, who participate on many conferences and are interested for new methodological issues. Their opinions do not mirror reality.

A methodological lecture for mathematics teachers of a small town in Hungary

The third trial is to be analysed in more detail. It is based on a presentation by the first author, in a small town in the southern part of Hungary for 25 teachers. The slides of the aforementioned presentation and a paper about our topics (Ambrus, 2015) were sent to the teachers who were also asked to study them and reflect on them by answering pre-formulated questions, that were grouped under corresponding issues (see Appendix C). Their responses are presented at the section of results in the present paper.

For studying the materials and answer the questions we asked experienced mathematics teachers from Budapest, the capital too, whose responses are also presented at the section of results. The comparison of the responses of the two groups of teachers forms essential part of the present study.

THEORETICAL BACKGROUND

BADDELEY'S MODEL OF MEMORY AND THE COGNITIVE LOAD THEORY

In this section, a summary of the theoretical base for the aforementioned presentation and paper (Ambrus, 2015)—the respondent teachers were provided with these materials—is presented. The application of the basic ideas of this theory—the cognitive load theory—to the practise of teaching mathematics, and in the field mathematics didactics at all, is fairly new, and is considered to be important and useful by the opinion of the authors, as they can give answers to some chief problems in mathematics learning and teaching, and specifically to problems of teaching mathematical problem solving.

The human cognitive architecture—Working Memory and Long Term Memory

“Any instructional procedure that ignores the structures that constitute human cognitive architecture is not likely to be effective.” (Kirschner et al., 2006)

Baddeley's model of the structure of human memory is widely accepted among neuroscientists. According to this model the parts of the structure are perceptual memory, working memory and long term memory (Baddeley et al., 2009). The latter two parts are to be analysed in more detail.

Working Memory (WM) is the 'work-bench' of our brain; it is the active problem space. It has four components: *phonological loop* to hold and rehears verbal information; *visual-spatial sketchpad* to hold and rehears visual and spatial information; *episodic buffer*, which connects the verbal and visual-spatial information, directed by the *central executive* with the help of the information taken from the long term memory. The central executive is the so called supervisory attention system, as it monitors and controls the information processing in our brain. Our WM constructs plans, uses transformation strategies, analogies, and metaphors, connects mental units during thinking, conducts abstractions and externalizes mental representations. WM has a very limited capacity of holding 7 ± 2 info units (Miller, 1956). Its time limit supposed to be 18 - 30 seconds without rehearsal. There is also a processing limit too: if we shall organize, contrast, compare, work on information, only two or three items of information can be processed parallel. (Baddeley et al., 2009)

Long Term Memory (LTM) contains information in the form of schemas. Schemas are abstract, structured, dynamic representations of information. Schema-automaticity means a skill, a learned procedure that is stored in LTM and is 'ready-made' available for WM, so that it does not place demand on WM to generate it. It has a very important consequence, that we may extend the capacity of WM with recalling a relevant schema from the LTM. It functions as only one information unit in the WM. (See capacity limit!) In WM, novel information is incorporated into existing schema(s), or similar schema(s) are produced and altered, or new schema(s) are recoded back into the LTM. A huge difference between experts and novices is that experts have a lot of solution schemas, which they can apply in traditional problem solving, while novices, that is most students do not have as many schemas. Additionally, there is no known limit of the capacity LTM or limit for the time duration of schemas being stored in LTM. (Baddeley et al., 2009)

The Cognitive Load Theory and the types of cognitive loads

Cognitive Load Theory was developed by John Sweller in the second half of the 1980s (Sweller, 1988). Cognitive load (CL) can be defined as the load imposed on working memory by information processing. The theory distinguishes three main types of cognitive loads, described in (Sweller et al., 1998).

The term *intrinsic cognitive load* (also sometimes called "essential processing") refers to the cognitive load imposed by the pieces of information that must be processed simultaneously, and that are directly connected to the problem itself. For example, when solving word problems, this information would derive for instance from reading the problem and understanding the text, conducting the mathematization process, doing the required operations within a mathematical model. Intrinsic cognitive load is embedded in the problem; teachers cannot really influence it by methodological tools that are independent from designing the problem itself. At solving complex tasks—mathematical problems—the intrinsic CL is very high.

Extraneous cognitive load refers to the cognitive load imposed by the manner information is presented. This may include unnecessary superfluous information (such as background music), holding mental representations of facts or figures, or separating related information (such as a geometric figure and related written statements). Extraneous cognitive load can make it much harder for the students to process information, and as it is not embedded in the problem, teachers can

influence it much more.

Germane cognitive load (also sometimes called “generative processing”) is the cognitive load placed on working memory by schema formation, integration, and automation. Germane cognitive load is decisive at mathematical problem solving. It may explain observed differences in students’ performance reflecting their relative experience, ability level, and content knowledge.

In summary, total cognitive load = intrinsic load + extraneous load + germane load. When planning the teaching process, teachers must take the potential total cognitive loads imposed by problem-solving and instruction methods into consideration, as too much cognitive load will probably impede learning.

Teachers can decrease the external cognitive load with conscious design of instruction, and the intrinsic CL can be handled, for instance, by dividing the problem into smaller parts.

Measuring cognitive load

There are three general methods for measuring cognitive load: subjective, physiological, and task- and performance-based. (Paas et al., 2003)

The *subjective method* of measuring cognitive load is highly realizable in the classroom. It is based on the assumption that students can access and express the mental effort they are expending. An often-used technique of subjectively measuring cognitive load is the one-dimensional ninth grade symmetrical category scale, developed by Paas (1992). In this technique, students rate their perceived mental effort after completing a problem on a nine-point rating scale (ranging from “very, very low mental effort” to “very, very high mental effort” (Paas, 1992)).

The physiological method of measuring cognitive load includes measuring heart rate or eye activity while students are solving problems.

Task-based and performance-based methods for measuring cognitive load consist of measuring primary task performance (actual task performance) and secondary task performance (based on a secondary task, performed concurrently with the primary task), using a relevant scale.

INSTRUCTIONAL DESIGNS TO REDUCE COGNITIVE LOAD

If cognitive load on working memory is to be limited for optimal learning, instructional methods to reduce and control cognitive load must be included in education, for instance and perhaps primarily at solving complex mathematical problems, in which case intrinsic cognitive load, and therefore total cognitive load are relatively high. Some instructional methods and principles that are relevant to reducing cognitive load are included in the followings.

The use of worked examples

One instructional way of reducing cognitive load is the use of *worked examples*. For the aim of promoting individual thinking of students, types and the amount of guidance are central issues within the problem solving tradition. Some researchers also call the attention that the almost complete lack of guidance may not be effective for every student: "Research has provided overwhelming evidence that, for everyone but experts, partial guidance during instruction is significantly less effective than full guidance." (Clark et al., 2012). Though Clark's standpoint may probably be quite exaggerated, it highlights the importance of the issue that the design of appropriate guidance is one important factor in planning problem solving. One way of doing so is using worked examples at certain phases of a course of problem-solving, for instance after the periods of students' individual trial and discovery. The use of "worked examples" is a technique in which the solution to a problem is explained in details to students by the teacher or peers. This allows students to concentrate on the essential problem states and possible related moves, as well as on the solution schema. It also facilitates students' integration of the solution schema into their long-term memory. Textbooks used in classrooms usually contain worked examples too, as well as problems that are to be solved individually by students. Even in one of the well-established and respected mathematics teaching methods in Hungary, in which individual thinking and discovery of students are central characteristics, in the so called Pósa method, the detailed discussion of the solution of problems, with the definitely expressed aim of preparing them to be hold in the LTM is an essential part.

Completion problems belong to a special form of worked examples. In such problems, there are gaps in the presented solution, and students are asked to fill them.

Open problems (usually called goal free problems too). For some problems, the dis-

tance between the starting phase and the goal is very high. With such problems, it is desirable to ask students to find all the relevant data they can find in the process of solving the problems. In our experiments, the “opening” of closed problems goes in this direction.

Split-attention effect, modality effect and redundancy effect

For different representations—visual and textual—of the same concept, procedure or strategy, students need to create their mental representations. This requires them to “split” their attention. This phenomenon is called the split-attention effect (Kalyuga et al., 1999, Paas et al., 2003). If representations of related information are very far from each other (physically), it may be difficult for them to integrate these representations into a single mental representation that will allow them to learn the most effectively. Therefore, in teaching, it is desirable to present visual and verbal information in a way that facilitates students’ integration of this information into one single mental representation. For example it may be desirable to write the equation of a function (physically) close to the function of a graph.

Modality effect refers to managing essential mental processing of different forms of information. It also explains why people learn better from a multimedia lesson when words are spoken, rather than printed, besides a picture. (Paas et al., 2003)

Redundancy effect occurs when multiple sources of the same information are presented, though it could be understood on the basis of one source, and the others may be disturbing. (Kalyuga et al., 1999).

THE IMPLICATIONS OF CLT FOR TEACHING MATHEMATICAL PROBLEM-SOLVING

One main goal of mathematics education is to enable students to be successful mathematics problem-solvers. Different fundamental positions on how to accomplish this goal can partly be distinguished on the basis of the type of instructional guidance.

In Hungary, a considerable number of mathematics educators have strongly been influenced by the ideas of G. Pólya, Z. Dienes, and T. Varga, and we believe in problem-based discovery learning to be a highly effective method of learning and teaching mathematics (Dienes, 1960; Halmos & Varga, 1978; Pólya, 1957;

Varga, 1965). However, when designing the appropriate type of discovery learning procedure, and making decisions on the role of guidance in the process, it is highly important to distinguish between the highly selected mathematically-gifted students and the less talented ones, for whom a different and perhaps stronger guidance may be more effective. Recent low results on the international and national tests of mathematical competences of the majority of Hungarian students support the necessity of bringing the less talented students into the focus of problem solving design. The international literature in didactics of mathematics and educational psychology present a variety of views on the application of studies on memory structure, cognitive architecture, and cognitive load theory, which can play a major role in planning problem solving for the majority of students in a given country. Researchers who have conducted scientifically controlled, randomised studies in this area, and whose findings we reviewed include John Sweller, John Hattie, Richard E. Clark and Paul A. Kirschner (Clark et al., 2012; Hattie et al., 2014; Kirschner, 2002; Sweller 1988, 2003; Sweller et al. 1985, 1998, 2010, 2011).

The examples Pólya used to demonstrate his problem-solving strategies are fascinating. ... It is possible to teach learners to use general strategies such as those suggested by Pólya (Schoenfeld, 1985) but that is insufficient. There is no body of research based on randomised, controlled experiments indicating that such teaching leads to better problem solving. (Clark et al, 2012).

Recommending partial or minimal guidance for novices was understandable back in the early 1960s ... We now are in a quite different environment; we know much more about the structures, functions, and characteristics of working memory and long-term memory, the relations between them, and their consequences for learning, problem solving, and critical thinking. We also have a good deal more experimental evidence as to what constitutes effective instruction: controlled experiments almost uniformly indicate that when dealing with novel information, learners should be explicitly shown all relevant information, including what to do and how to do it. (Clark et al, 2012)

Based on (Clark, 2012) we claim for the importance of incorporating guided discussions, and the use of worked examples into the problem solving process, which seem to be present for instance in the Pósa method too, for instance as recurring phases between periods of individual thinking and discovery, making

also those relevant information available to students which they could not find themselves during their discovery.

Raising the effectiveness of learning during problem solving

The superiority of chess masters comes not only from having acquired clever, sophisticated, general problem-solving strategies but rather from having stored innumerable configurations and the best moves associated with each in long-term memory. De Groot's results have been replicated in a variety of educationally relevant fields, including mathematics (Sweller & Cooper, 1985). ...“Mathematical problem-solving skill is acquired through a large number of specific mathematical problem-solving strategies relevant to particular problems. There are no separate, general problem-solving strategies that can be learned. How do people solve problems that they have not previously encountered? Most employ a version of means-ends analysis in which differences between a current problem-state and goal-state are identified and problem-solving operators are found to reduce those differences”. (Sweller et al., 2011)

If the schema corresponding to the problem to be solved is not available in the student's LTM, the student must search for, or construct a relevant solution process with a highly loading use of the working memory. The means-ends analysis technique is a strategy to control such a problem-solving search. Given the difference between a current state and a goal state, an action is chosen to reduce that difference. The action is performed on the current state to produce a new one, and the process is recursively applied to this new state and the goal state. This search in means-ends analysis causes a heavy burden for working memory. Preferably the result of this procedure is the creation of a new solution schema in the LTM. However, if nothing happens in the long-term memory, there will be no learning. Using worked examples as complete means to the means-end analysis enables students to concentrate more on problem states and possible solution steps, and to transfer solution schema into LTM for later retrieval.

The “class teaching” method

In Hungarian mathematics education, the “class teaching” method is dominant, as well as the tradition of using the so-called “problem-oriented” style. However, the effectiveness of this method and style are not proven. “In real classrooms,

several problems occur when different kinds of minimally guided instructions are used. First, often only the brightest and most well-prepared students may disengage. Second, others may copy whatever the brightest students are doing—either way, they are not actually discovering anything. Third, some students believe they have discovered the correct information or solution, but they are mistaken and so they may learn a misconception that can interfere with later learning and problem solving. Even after being shown the right answer, more if in not in the most proper way, a student is likely to recall his or her discovery—not the correction. Fourth, even in the unlikely event that a problem or project is devised and all students succeed in completing minimally guided instruction may be much less efficient than explicit guidance. “What can be taught directly in a 25-minute demonstration and discussion, followed by 15 minutes of independent practice with corrective feedback by a teacher, may take several class periods to learn via minimally guided projects and/or problem solving.” (Clark et al., 2012)

To summarize, the expert problem solvers differs from the novice ones in that the formers can use their existing solution schemas stored in LTM to solve new problems. For the latter ones, and for most students, a more guided instruction, more precisely inserting more guided parts can be more effective. If they also study worked examples, they may learn the solution strategies and construct the desired new schemas more effectively. If the students already have the relevant schemas, the use of worked examples may be redundant and disturbing, that is called the expertise reversal effect (Kalyuga, 2007).

METHODOLOGY

The first author asked the participants at the start of his aforementioned presentation about the level of Hungarian mathematics teaching. Most of the teachers reacted that it is world famous. As arguments they named some famous mathematicians with Hungarian origin, and noticed that more and more Hungarian students continue their studies on famous universities abroad. Telling them the Hungarian PISA 2015 results – 40th place in mathematics – they were not convinced yet. Only presenting them the mathematics test results at the Technical University Budapest and at Eötvös Lóránd University Budapest at the start of students' studies – 30-40% of students were successful on the first trial – though the tasks required mainly basic mathematical knowledge, they started to accept: there are problems in Hungarian mathematics teaching.

Our main aim was to discuss with the teachers such kinds of new ideas which may help to reach more students, the average students too. As a first trial, we asked teachers from a small town in South Hungary. We have had the possibility to ask some young teachers – 10 years of experience – from the capital. It is interesting to compare the different views between small town teachers versus teachers from elite schools of the capital.

The first author was invited to give the aforementioned presentation for mathematics teachers of upper-primary and secondary schools in the beginning of May 2016. The only wish of the organizers was that the speech should not be only about 'nice' mathematical problems, as they were interested in possible answers to problems of everyday mathematics teaching. Beside the questions of 'what' and 'how', they were interested in the question of 'why' too." 25 practicing mathematics teachers participated on the meeting. (10 secondary school mathematics teachers of grades 7-12, 15 middle school mathematics teachers of grades 5-8) The topics of the talk included a model of the *structure of human memory*, the *cognitive load theory*, reducing the cognitive load, avoiding the *split-attention effect*, facilitating *schema automation*, the use of *worked examples*, the similarities between *mathematics problem-solving* and playing chess, and the efficiency of *distributed practice*. At the end of the presentation, we asked the participants to react on it, by answering some questions, primarily in connection with the issues of automaticity, cognitive load, worked examples, mathematical problem-solving, the split-attention effect, and distributed practice. Each participant received the presentation slides of the talk. Unfortunately we received only 1 response, the reason for which might be that teachers are generally quite busy with a high teaching load—22–26 lessons, as it was already mentioned—per week, and responding was optional, opposed to the great amount of their compulsory administrative duties in their schools.

In middle August, we have sent the presentation slides again, and additionally the first author's article (Ambrus, 2015) on the same topics, that forms an adequate summary of the main parts of the talk. To make it easier to comment on the topics, we formulated clear questions for the teachers, but they had the possibility to express their opinions relating other problems too. As a result, 1 secondary school teacher replied, and a community of mathematics teachers (CMT) of a secondary school discussed the questions and have sent us their quite short answers, as the collective opinion of them.

For further investigations, we have sent the materials to 3 more experienced

mathematics teachers who work also with mathematically talented students, and to 4 young mathematics teachers—each having approximately 10 years teaching experience—who have also started their PhD studies in mathematics-didactics but continuing their teaching too. Unfortunately only 1 out of the 3 experienced teachers responded. The 4 young teachers all answered the questions, based on their personal teaching experiences.

Altogether we received the responses of 3 individual practising teachers (T - 1, T - 2, and T - 3) and 1 group (CMT) from outside the capital of Hungary, and of 4 young teachers who started their PhD study in Didactics of Mathematics (PhD S - 1, PhD S - 2, PhD S - 3 and PhD S - 4) from the capital.

RESULTS — TEACHERS' RESPONSES

The limited capacity of working memory, reduction of the cognitive load (Issue 1)

As was expected, this question was very new and unusual for the teachers. Most of them focused on the '*what to teach*' and '*how to check it*' questions. Laurinda Brown (University of Bristol) visited Hungary and Hungarian secondary schools several times. She summarized her opinion in the following: "You in Hungary teach mathematics, we in England teach children." We experienced the same (for Hungarians) in our study.

The teachers from the countryside summarized their general opinion about the studied issues: "We can't take the cognitive load theory into consideration, because the obligatory content material is too much, and the teaching time is not enough. We can focus solely on the matura exam."

The answers of the young teachers differed from that of the teachers from the countryside. The main reason may be that they teach gifted and diligent students, in elite schools. Another reason may be that they started their PhD studies, besides school teaching, had learnt about the cognitive load theory before the survey. Their answers may mirror the efficiency of their previous studies.

To some extent, I have often been paying attention to most of the following teaching techniques, though before reading the article it was done rather intuitively, and now I will more consciously use them in order for the reduction of the extraneous cognitive load and the increase in germane cognitive load.

The board shall be clean at the beginning of lessons, as potentially interfering pieces of information from other lessons may increase extraneous cognitive load.

The decision of whether or not to remove pieces of information on board from different previous segments of the lesson shall be a matter of conscious and continuously monitored process. Previously put on board, but relevant pieces of information, together with the one in focus at a particular moment may support the formation of schemas at a higher level of abstraction. However, if the information is not relevant, it may generate the split-attention effect.

New topics shall be on a new page in students' exercise books, if some relevant pieces of information from the previous topics are not needed to support the increase in germane cognitive load.

Basic ideas, important segments shall be highlighted by marking them with a circle, the use of coloured chalk / pencil, or by other means. However, when using coloured pencils, students need to change the pencils physically, which may draw their attention from the problem itself, and may generate the split-attention effect.

...

Ambient noise is to be minimized. However, something that is ambient noise, that is part of the extraneous cognitive load for one student can be an important piece of information and part of the intrinsic or germane cognitive load for another student. (PhD. 1)

In order for reducing the extraneous cognitive load, I consider the creation of calm and, depending on the situation, relatively silent atmosphere to be the most important. I do not mean an overdisciplined state of being in the classroom, only the concentration of 'energies' to the intrinsic and, to the extent it is possible, to germane load. When trying to maximize silence, that is, minimize noise, it is important to note, that during pair or group work, noise is relative, loud talk of students may not be considered to be part of extraneous load, at least not for those talking, therefore I usually let them talk freely. (PhD S – 2)

The notion of cognitive load was completely new to me, and I consider it to be very useful. Intuitively, I have always tried to reduce the split-attention effect during my lessons. However, after reading the article, it has become obvious to me that I shall attempt cons-

ciously to have students' cognitive load reduced, that is, I need to pay (more) attention to this criterion not only during the lessons, but also in the lesson planning phase. I am a "speaking too much" kind of teacher. For example, I used to give supportive instructions during (usually in the middle of) individual problem-solving procedures, instructions that I considered necessary to help the students find the desired solutions. From now on, I need to pay attention to give every piece of supportive instruction in the beginning of the procedure, before students start solving the difficult problems.

During the lessons, it can (relatively easily) be noticed, because of the way they look, who is paying attention. I consciously monitor students, and I try to get them involved in the lesson, as much as possible, by asking questions or asking them solve problems at the board. We usually listen to several solutions to a problem and opinions of several students, in order for assisting the maintenance of attention, as well as for the discovery (and correction) of otherwise hidden mistakes. The results of these deep discussions on problems are often tested in the next lessons. (PhD S - 3)

Schema building, schema automation (Issue 2)

It is a hard task to convince this age group (teenagers) about the necessity of learning. In my opinion, without the knowledge of mathematical definitions and theorems, it is impossible to go further in mathematics. However, most of the students use only formula collections, without memorizing the required definitions and theorems. (T - 1)

It is necessary to change the beliefs of many students that we do not need to learn, only to understand mathematics. Just the opposite of it is true! Of course, it is easier to learn something if we understand it. Since I experienced that the students do not learn at home, I asked the students to explain the studied material at the mathematics lessons. I recommended they repeat the material with the aim of fixing it in their memory. Concerning the basic mathematical procedures, each student in the class had to demonstrate at the blackboard that he /she can apply them. (T - 2)

For example, in case of the definitions of the trigonometric and the square root functions, and the identity rules of operations with powers, I usually ask all my students to memorize not only the formulae, but the textual forms too, and assessment is regular at the beginning of the lessons. I think memorizing fosters schema automation. (PhD S - 4)

“We try to teach the procedures that are solutions to different types of problems, as well as to practice and control them in ‘weaker classes’, focusing strongly on the requirements of the mathematics matura examination.” (CMT)

Students’ learning the appropriate definitions, theorems and procedures is a prerequisite of the creation and automation of the schemas. I consider it useful for the students to understand and learn the proofs of some selected theorems, as it helps them to have an overall view of the structure, the connections within a topic, and between the different topics too.

I regard learning literally the definitions of concepts highly important, because students can only solve problems if they know exactly the ‘meaning’ of what is being done. The accurate knowledge of definitions is also important during solving special tasks designed for investigating the borderline cases in connection with a concept. The notion of logarithm is a good example for this. Those and only those students were successful in solving tasks with logarithms in my classes, who, at the beginning, had learned and understood the concept. Even the simplest tasks with logarithm usually fail to be solved by those students who do not know that they are manipulating with exponents of powers. (PhD S - 3)

Teaching mathematical problem-solving (Issue 3)

Problem-solving is similar to the chess game. A successful problem-solver needs to know 3000-3500 problem situations with their solution steps. I usually say to my students that they need to have a database in their memory, which contains concepts (definitions), theorems, procedures and solution ideas. They need to have a search engine which compares the elements of the problem situations with

the elements of their database. (T - 3)

I do not teach mathematical problem-solving, as students are taught by the problems themselves. I only assist and facilitate the process by selecting the appropriate problems in the order which enables students discover at least parts of the solutions, as they can use the results of previous problems. Individual work is of utmost importance. If the students are less talented, the problems should be less difficult. Curriculum should only partly govern the selection of the level of problems. The ultimate learning objective, in my view is the development of the ability of conscious schema creation, instead of the creation of schemas itself; in long term, the development of this ability results in the development of a greater set of schemas. If there are general problem-solving strategies, they are better learnt by concentrating on specific problems, which I do in my teaching practice. (PhD – 1)

The basis of effective problem solving is to know a great many types of problem solutions with their solution steps. It is a long term goal of mathematics teaching. We use the discussion of homework to make the problem situations and the applied strategies and methods conscious. (PhD S - 3)

Use of worked examples (Issue 4)

Most teachers understood worked examples as they find it in the textbooks, where only one problem is worked out and it is to be followed by individual problem solving, though it is more than presenting only one solution.

We have only 3, 4 or 5 mathematics lessons a week, hence, there is not enough time to discover everything. Based on my more than 30 years of mathematics teaching experience, I hold that if we want our students to acquire a mathematical idea, such as a new concept or procedure effectively, they need to be concerned with them (at least) three different times. First we present it and help the students to understand and apply the idea. Second, we need to refresh the learnt ideas by review. Third, the idea shall be embedded in a complex problem situation. (T - 3)

“The use of worked examples is necessary in mathematics lessons.

In my classes, the textbook examples are to be learnt at home, and during the lessons we solve similar tasks.” (T - 1)

Worked examples of the textbook might be a great help for the teachers. Worked examples should be memorized and students should be tested on them. In this case, the students are supposed to work with them. It is a pity that students do not check their notes on the solution of the tasks at the lessons, as well as that they do not revise it at home. They write down in their exercise book only what the teacher is writing on the blackboard, and they don't fix what the teacher is saying, so that part is usually forgotten. Sometimes they write down the solution to a problem without knowing what the problem was, as they only copy from the blackboard. (T - 2)

I start every topic with a presentation and discussion of worked examples, as the majority of students would not discover the solutions for new kinds of tasks without detailed guidance, or it would take too much time for them. First, I present the students with 3-4 not too difficult worked examples, and then they work independently on 1-2 similar tasks. Then we repeat this process with more and more difficult tasks. At the end of the topic, for 1-2 lessons, students work individually again, meanwhile I help them one by one, and monitor the common difficulties, which are then discussed together again.

I also noticed that students remember their solutions more correctly, the ones they discovered individually. However, individual discovery happens only when they already have some level of experience in the topic and the task is easy or moderately difficult. I don't think the method of discovery learning is useful when introducing new topics, as it is highly time-consuming. Discovery learning is more advantageous when students have an extended set of schemas stored in their LTM. If still there is a student who has an idea, they can present their solutions at the board, and we, others may help, with my guidance. In that case we “discover” together the solution to the problem. (PhD S - 3)

The role of distributed practice in mathematical problem-solving teaching

“I liked to hear that five times 10 minutes practice is more effective than one piece of 50 minutes long practice.” (T - 2)

The curriculum is very demanding, there is too much to teach, and time is not enough to achieve the matura examination requirements, so we always have to hurry up. Because of the few numbers of mathematics lessons per week, we have very little time (if any) for revision. Nevertheless, we have been convinced that we should take into consideration the limits of working memory. We really hope in the change. (CMT)

Topical reviews of previous lessons have always been part of my lessons. I try to divide the topics into smaller parts and after being reviewed, it is followed by an assessment. At the end of the textbook chapters (topics) we always have larger reviews. At the start and at the end of the school years we do revise too. Unfortunately, it does not work effectively because of the decreasing number of lessons per week. Another problem is that there are administrative controls, where the students' exercise books, the class registers and the syllabus are compared, and they must be congruent to each other. Although I often see that some concepts and procedures are not effectively acquired by my students, but we must go further, there is no time for extra practice. (T - 1)

The syllabus I (we) use is partly based on the spiral principle of learning mathematics, therefore distributed practice is ensured year by year. More regular revision is really difficult, though one time-saving way of revision, starting each lesson with lightning questions about previously learnt material is really preferred by me. Besides these techniques, I of course plan revision at the end of each topic, where students have the opportunity to discover and understand connections between mathematical objects at a higher level, and therefore may formulate new schemas. (PhD S – 1)

Before we end a chapter, it is necessary to review, but usually we also start each lesson with review questions, with the aim that the students memorize and ‘experience’ the important concepts and ideas. Checking the homework belongs to this review. If you want

to go further effectively, you need to refresh the ideas of the previous lesson(s). (PhD S - 4)

Based on the conclusions may be drawn from the presented figures of forgetting curves, the optimal frequency of revision would be revising the same material every week or even every lesson (for a certain period). However, the recent years, due to the narrow time frame, I could only try to have a one week or two weeks long revisions at the end of each school year. I gave students 'summer review tasks', in order for a more effective start in September.

This year, as a consequence of my studies and experience of how quickly students' knowledge sinks into oblivion, I decided to insert a revision lesson after each topic with assorted types of exercises. The study of the forgetting curves presented in the article has secured my conviction that there is no sense in teaching a new topic without remembering the previous ones. Solving complex problems with elements from different topics may also be an effective and less boring form of revision.

In conclusion, after reading the article, I realized that I need to be much more conscious about planned revision. (PhD S - 3)

DISCUSSION

Some preliminary remarks are the following.

- The discussed topic of implementing CLT into the praxis of mathematics teachers is relatively new in mathematics didactics literature.
- The present paper is based only on a few simple case studies. Therefore, it is not possible to conclude general statements from them. The intention of the present study is to place emphasis on the importance of, and hopefully conduce to the development and design of in-service mathematics teacher courses, by contributing to the introduction of applying CLT in planning pre- and in-service teacher training. The discussion of every issue raised in the following sections could and should be elaborated in further studies.
- Teaching mathematical problem solving is a complex phenomenon. It is more than solely giving nice problems to students and leaving them to solve those problems. The design of teacher assistance during problem-solving is of high importance. We are convinced that notions of cognitive

load, schema automation, use of worked examples and distributed practice are all significant in planning effective problem solving.

Lack of reflective teaching

At a small conference in Budapest, a principal emphasized in her talk, that the teachers in her school and in general, the majority of Hungarian teachers are not able to, not trained to reflect on their teaching methods and practise effectively. It is in agreement with two pieces of results of our questionnaire. First, in spite of the considerable number of participants of our talk, we received an extremely few number of responses. Second, the answers of the community of mathematics teachers (CMT) were highly narrow. Both pre-service and in-service teacher training programs need to consider how to develop teachers' ability to reflect on their teaching.

Considering the low number of teachers' responses, it could also be possible that many teachers did not understand the theory of cognitive load sufficiently, so that the majority could not see enough relations to their teaching practice, in which case, further and more detailed lectures and seminars on CLT and its implementation into classroom teaching would produce different, more desirable results.

Lack of background knowledge about CLT and its implementation into teaching mathematics

Cognitive load theory and the corresponding notions and phenomena, such as the (limited) capacity of working memory and split-attention effect have appeared to be, at least partly new for all our responding teachers. Although some of them intuitively used techniques to reduce cognitive load before the presentation and before reading the article, they all confessed not to be conscious and consistent about using them. However, the majority of them have found the theory useful and worthy to be applied in teaching problem-solving and in their teaching practise in general. Getting to hear, read and have discussions about the theory has directed their attention to problems which they consider to be essential in teaching and learning efficiency, but which have been hidden so far. Therefore, our present experience concerning practising teachers' up-to-date professional knowledge, and those similar to ours may be helpful in the organizations of teacher meetings and choosing the relevant content material for in-service teacher trainings.

Difference between the answers of teachers from a small town and the responses of teachers from the capital

The responses of the teachers from the small town and its region seem to reflect a much narrower way of thinking concerning their students' learning and their teaching process, than that of the teachers from so called elite schools of Budapest who are also conducting their PhD studies. Although, the latter ones may have a much broader vocabulary allowing them to express themselves more clearly and in more details, there still seem to be difference in the range of ideas as reflections on the suggested issues as well, not only in the way of the linguistic formation of these ideas. For instance, the interpretations of the role, and therefore partly the concept of worked example by young teachers from capital and by the teachers from the small town show a great difference. While the latter mainly claimed to use worked examples only as introductions to new concepts and types of problems, the PhD students tend to apply worked examples in a more complex way, allowing their students to take part in the alternating phases of individual problem-solving and discussion on worked examples.

The responses of the PhD students reflect interest in more complex questions of teaching, students' cognitive development and learning, and the relationship between the two, which is of course also supported by the fact they have decided to continue their studies and they wish to conduct research studies as well. In connection with a potential reconstruction of the in-service teacher training system of Hungary, it also seems to be important to take into consideration that they probably need a different type of in-service training after their PhD studies. Consequently, in-service teacher training may not be a unique, 'one way' system, but should have many different segments, with different types of approaches to training based on participants' schools local specialities and their qualifications.

Confidence in teacher - researcher relationship

The relationship between practising teachers and mathematics educational researchers, as any kind of fruitful, effective relationship, must be based on trust. When a teacher trainer works with teachers, the trainer first needs to win their trust, as only in this case will they express their real opinion. It is supported by the fact, that among the participants of the talk those and only those teachers responded who have been or were students of the first author, that is they had already

had an established relationship, and mutual confidence at the time of responding.

However, another reason for the difference in the willingness to respond may also be that those who had already had an established relationship had also already had a better understanding of the intention of the authors. It further supports the claim for more lectures and seminars with carefully chosen lecturers and seminars on CLT and its implementation into classroom teaching.

Assessment-oriented teaching

We have known the phrase 'what you test is what you get' for years, or as Alan H. Schoenfeld noted in his paper on mathematical proficiency and assessment that "teachers feel pressured to teach to the test" (Schoenfeld, 2007). The present study revealed that it may be the case in Hungarian secondary education too. A large proportion of the responding teachers emphasised that the requirements of the matura examination have a crucial, a primary influence on their teaching practise, on planning the teaching process and on the selection of problems. They always struggle with time. They have less and less time for teaching problem-solving, as well as for using methods that take into consideration the results of cognitive load theory and corresponding ideas from cognitive psychology, such as distributed practise.

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APPENDIX A

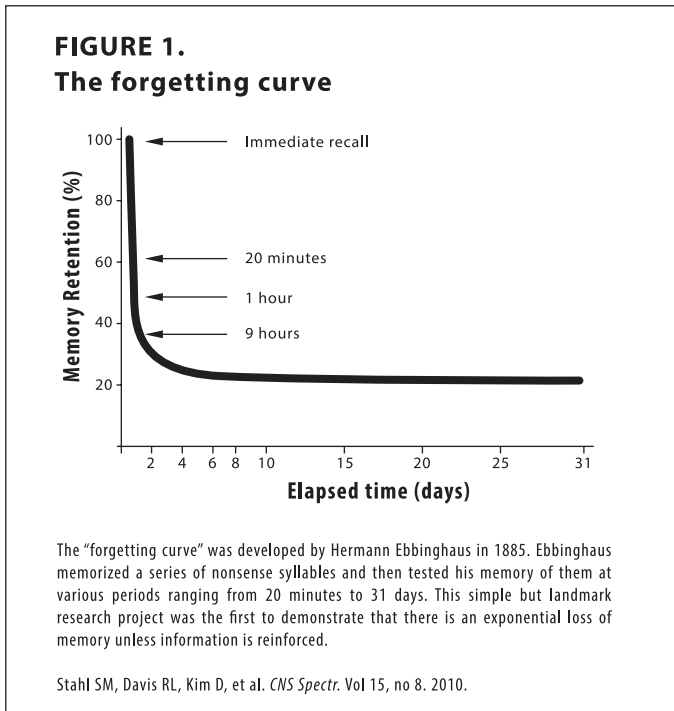


FIGURE 1. Forgetting curve without distributed practice (Stahl et al., 2010)

APPENDIX B

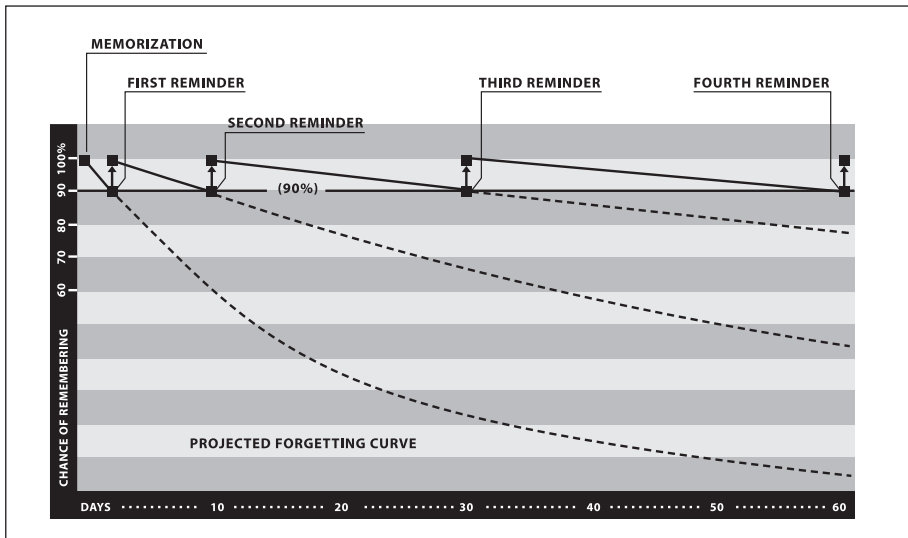


FIGURE 2. Forgetting curve with distributed practice (Stahl et al., 2010)

APPENDIX C

The questions—grouped under issues

The issues, and the corresponding questions, were presented together with short notes and explanations on the basic concepts used in them, to avoid misunderstanding. The terminology in CLT was, and perhaps is very new and hard to follow for practicing teachers, for this reason we tried to formulate the questions as simple as possible using terminology close to everyday teaching practice.

Issue 1

The working memory has very limited capacity and a strong time limit. It may hold 7 ± 2 information unit (Miller, 1956), and it may hold the pieces of information without rehearsal for about 20-30 seconds. At parallel processing, most people can do only 2 processes. Students may be overloaded in the lessons, if too

much information should be worked out in a short time. (Baddeley et al., 2009)

Can you take this problem into consideration in your teaching?

What kind of methods do you use (have you been using any method) for avoiding the split-attention effect and reducing the cognitive load?

Issue 2

There is a strong link between *Working Memory* (WM) and *Long term memory* (LTM). If the WM can activate the relevant piece of information stored in LTM in the form of a schema, the processing of this piece of information doesn't need extra WM capacity (Sweller, 2003). For this reason, schema automation, and *automaticity* are central issues in mathematics teaching too, as schema automation expands the WM capacity.

What do you think the role of schema automation is in your teaching practise?

Issue 3

An experienced problem solver in any domain has constructed and stored huge numbers of schemas in long-term memory that allow problems in that domain to be categorized according to their solution moves. In short, the research suggests that we can teach aspiring mathematicians to be effective problem solvers only by providing them with a large store of domain-specific schemas. (Sweller et al., 2011)

How do you teach mathematical problem-solving?

Issue 4

When introducing new concepts, types of problems or theorems for study, it may be advisable to give a definite time limit for discovery, and at a certain point use direct instructions, otherwise students can easily be overloaded (*cognitive overload*) because of the intensive search processes of problem-solving. After a limited period of trial, discovery, and discussion, it is better if clear solution processes, well-structured *worked examples* are presented to the whole class, so that students can concentrate on the main steps of the *solution schemas*, and so they are transferred to Long Term Memory with a higher probability. (Clark et al., 2012)

Do you have similar experiences in your teaching practice?

How often and when do you use worked examples?

Issue 5

Research on and the analysis of *forgetting curves* and distributed practice, also called spaced repetition (Stahl et al., 2010), leads us to the conclusion that regular revision has an essential role in effective learning, e.g. in learning via problem-solving. The respondent teachers studied figures of forgetting curves from (Stahl et al., 2010)—see Appendix A and appendix B—and were asked to comment on them from the point of view of their teaching practice.

How can you realize the implications derived from the analysis of forgetting curves and the idea of distributed practice in your own mathematics teaching practice?