OPEN TASKS IN MATHEMATICS: EXPERIENCES WITH ONE PROBLEM FIELD

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Abstract

In this paper, the task "Number Triangle" stands for problem fields. The problem field in question is planned by the author to be used in the Finnish comprehensive school. With its aid, our purpose is to show how open teaching acts in practice. The most important in these problem tasks is the way they are presented in teaching: A problem field should be offered to pupils little by little. And the continuation in the problem field depends always on pupils' answers. The answers of problems are not given here, since they are not as important as pupils' independent solving of problems. How far the teacher proceeds with the problem field in question depends on answers given by pupils.

Key words: *problem solving, open problems, problem fields, number triangle*

INTRODUCTION

Mathematics teachers have traditionally emphasized the goal "to achieve certain calculation skills" and thought that the other goals will then follow automatically. But research (e.g. Kupari, 1999) has shown in Finland that all pupils have not reached the goals set by the teacher, and there are some of pupils still struggling with the basic topics. Thus the other goals of mathematics teaching will be neglected.

According to the Finnish curricula (NBE, 2004), mathematics should be taught to pupils from Kindergarten to university with such a method that promotes their understanding of mathematical concepts and procedures as well as solving of problems. The very same wording can be found also in the new curriculum (NBE, 2014).

For most teachers it is still an open question how this goal could be reached in school teaching. But if problem solving situations are organized for all pupils, they will have opportunities e.g. to invent, to reason, to generalize, to make hypothesis, to probe the hypothesis, etc., and thus offer to all pupils a realistic possibility to reach these goals. One method to implement all this is the use of open tasks in teaching.

Open tasks

A task is said to be *open*, if the starting and/or goal situation are not exactly given (cf. Pehkonen, 2004). In such tasks, pupils have more freedom to work, since they can themselves determine some parameters for the task. And therefore, they will reach different, but equally valued correct solutions, since the result depends on the solvers' additional hypotheses and emphases. Thus open tasks can usually have several correct solutions or no solution at all. In such a problem field pupils can put for themselves problems, solve them and check their sensibility.

Thus a teacher's role when working with an open problem is – as Schoenfeld (1985) says – to act as a moderator of discussion and as a secretary. I.e. he/she will write all alternatives of solutions presented in the teaching group on the blackboard, without commenting them, also possible wrong tracks. Pupils' task is then to ponder the sensibility and the correctness of different solution alternatives (also possible wrong tracks), and they should discuss together on these alternatives. In traditional teaching, it is too much emphasized the role of "correct answers". Instead of answers, it is more important to develop and to clarify the solving method than the product, and to emphasize the pupils' own ideas for solving the problem.

Reaching for goals

When we change in mathematics teaching the emphasis from calculation skills to thinking skills, we can build mathematics teaching into such a frame that responses every pupils' needs. For example, logical thinking and creativity are needed in all areas of the life. It is something that is expected from cultivated citizens. Since our main goal is the development of mathematical thinking, we can choose the emphases of the teaching topics according to this viewpoint. We can focus on such contents where the development of mathematical thinking and creativity can be easiest implemented.

It is important to notice that this does not mean the abandonment of practicing of calculation routines. In the case of routines it is more a change of attitude. Some thirty years ago Wittmann (1984) has presented the idea that routine practices should be built in such a set of tasks that has a certain structure. Then we can, within this structure – along with routine practice – also develop the so-called higher order thinking. More details on the theoretical framework can be found in the publication of Pehkonen (2016).

In order to enlighten these ideas, there is depicted a problem field (Number Triangle) in the following. Following Zimmermann's ideas (2016), its implementation is explained with the help of a successful school lesson, and as such it is discussed in detail. The description of the lesson has been published earlier (cf. Pehkonen 1988), but only in Finnish. Furthermore, I have collected more experiences of this problem field during almost thirty years, in different surroundings.

Number Triangle

The problem field "Number Triangle" can be used from the elementary school to the teacher education, since it does not contain only calculations, but also reasoning. In the obligatory school (grades 1–9), solutions for the problem can be found by using experimenting method, but the general solution will lead to three equations and four unknown (e.g. in the upper secondary school). In order to reach a proper level of challenge in teacher education, one can ask teacher students, not only to solve the problem, but also to ask how they think that pupils in comprehensive school would solve the problem.

In autumn 1987 I experimented the number triangle as a separate problem in teaching group of 16 pupils in grade 7 in Helsinki. My original purpose was to practice and brush up mental calculations with natural numbers in a slightly unusual way. But from number triangles it was developed an interesting problem field within which we also could practice mental calculations with negative numbers. And the calculation with integers was one of the main topics in grade 7. The pupils were eager to search solutions for separate problems as well as to look for general solution principles. If the problem was not dealt with within some lessons, the pupils were asking when we will continue and wanted to explain their new solutions.

In the following I will describe briefly different phases of dealing with the number triangle chronologically. It is good to point out that all the lessons were not sequential, and that the time used for each lesson varied 10–15 min.

The 1. lesson

At the end of one lesson there was about 10 min. teaching time left, and I presented the Number Triangle problem (Fig. 1) to the pupils using the overhead projector. The problem was as follows: "Invent numbers in the circles of the triangle in the way that the sum in each side row is the same." I asked them to write it on their notebooks, since the task will be their next homework. Additionally we discussed what is the meaning of the wording "the sum in each side row is the same".

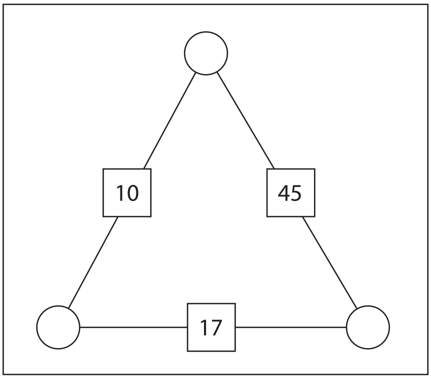


FIG. 1. The number triangle: Invent numbers in the circles of the triangle in the way that the sum in each side row is the same.

The 2. lesson

When checking the homework, I asked the pupils to draw their solution on the blackboard. But the pupils were wondering that they have different solutions, and asked me which might be the correct one. Then I asked three pupils to present their solution on the blackboard. And beforehand I asked what their row sums were, in order I would get three different solutions on the blackboard. In this connection we used the concept "row sum" with the meaning of the constant side row sum.

During the lesson I asked the pupils to guess, how many different number triangles there might be. One pupil stated carefully that, perhaps, there would be found still one more. Thus I gave them for the next homework to invent one different solution more.

In one lesson (where the topic was addition and subtraction with integers) I gave to the pupils an additional task: They should find out if it was possible to use negative numbers in the corner circles of the triangle. Some high-attainers developed very quickly one number triangle with a negative number in one of the circles. Therefore, my next question was “Can we use negative numbers in two circles?” that gave them something to ponder for a while.

The 3. lesson

In the beginning I asked the pupils to show three new number triangles on the blackboard. Then I asked again, how many different number triangles there might be altogether. A couple of the pupils had developed interesting methods to answer the question.

For instance, Ann looked at the number triangles on the blackboard (cf. Fig. 2) and stated that there are as many number triangles as we want. When I asked reasons for her response, she said that (in Fig. 2) the right-hand side is got from the left-hand side by adding in each “circle” the number 2. Therefore, we can get as many different number triangles as we want.

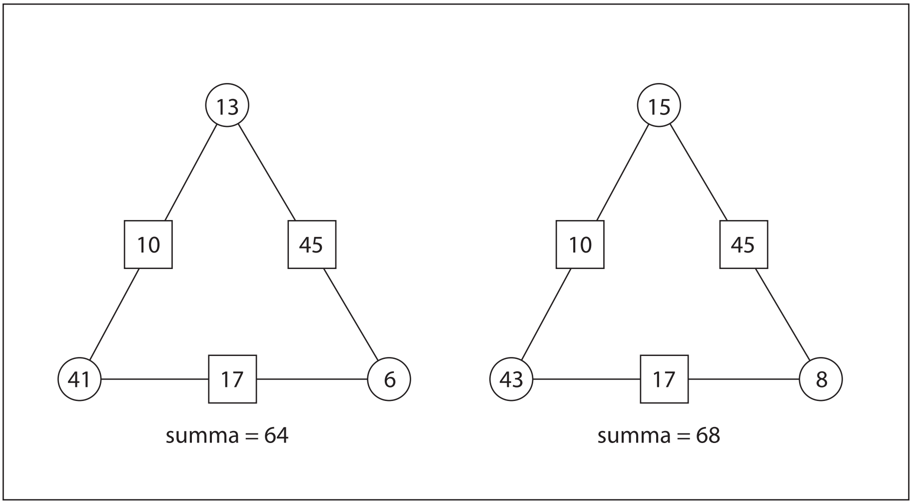


Fig. 2. Two examples of the number triangle.

Basil had developed at home an original method to produce different solutions. His principle was, as follows: Let us put into the upper circle an arbitrary number (e.g. 22). Because 10+22 = 32, we will look for a number into the right-hand corner a number, which will add with 17 the result 32 (thus 15). Since the right-hand row of the number triangle gives 22+45+15 = 82, we should look for a number in the left-hand corner that gives the row sum 82 (thus 50). But this method seemed to be so complicated that it should be explained twice (in both cases a different pupil explaining the method at the backboard) with examples, before most of the pupils said that they understood it.

In order to test how well the pupils have understood the methods explained, I asked from the teaching group, whether it was possible to construct a number triangle where the row sum is zero. After a while of pondering many responded that the task was impossible to solve. Thus I left it as a homework in the following form: "Investigate whether there exists a number triangle with the row sum 0. If not, what is the smallest positive number that could exist as a row sum?"

The 4. lesson

When checking homework it came out that only one pupil was able to produce a number triangle with the row sum 0. All other suggested different one-digit numbers as the smallest row sum. Since during the last lesson we had the adding method presented by Ann, I was expecting that the pupils would be able – at least some of the pupils – to use it in an inverse way (subtract the same number from each corner number) and thus get the row sum zero. But the pupil in question had found his solution by experimenting. Evidently they had not "melted" the method exposed by Ann – not even Ann herself.

For the next lesson I gave the task of producing a number triangle with the row sum 10.

The 5. lesson

Again a couple of pupils had found a solution. I used one whole lesson when trying to lead the pupils to think in Ann's way: We began with a number triangle where the row sum was 10. I asked how much it must be added into the corner number, in order we would reach the row sum 12. A pupil suggested that into each corner it must added number 2 – the others did not object. When I had drawn on the blackboard the number triangle in question, I asked them to calculate the row sum of each side. Quickly the first ones stated that it is 14. A pupil (Cecilia) suggested the following correction: Since the row sum increased with two (12–10 = 2), the difference should be divided by two and the resulted number (1) be added into each corner number. Thus we did and resulted the number triangle searched for.

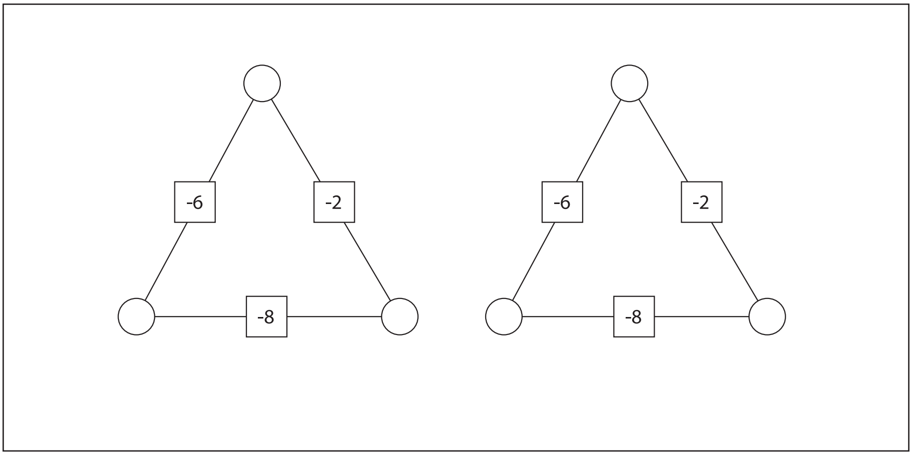
In order to test how many pupils had understood the general method, I asked them to produce a number triangle with the row sum 20. When asking what should be added into the corner numbers of the original triangle (the row sum 10), I received two kinds of answers: 10 or 5. The pupils were spontaneously arguing with each other. Only two or three pupils seemed to understand the general principle and to be able to use it. We drew both number triangles on the blackboard and solved the question by calculating their row sums.

My next question was, whether we can thus also decrease the row sum. And received a positive answer. Thus I put forward a task to produce a number triangle with the row sum -10. Together we observed that the row sum will decease by 20 from the original (the row sum 10), and therefore, from all corner numbers it should be subtracted 10. Since the lesson ended before all pupils were ready with their number triangle and have checked them, the task was left as homework. The second task for the pupils was to construct a number triangle with the row sum -20.

The 6. lesson

When beginning with checking of homework (where was also the number triangle task mentioned), I could hear from the teaching group a following bored comment "Do we have again that dull number triangle?" And immediately I reacted with saying that this was going to be the last lesson with it, although I have thought some more ideas of continuation.

Written exam



In the next written exam that was some weeks after the last number triangle lesson, I added into the exam the task (i.e. task 6) that is given in Fig. 3.

Fig. 3. The number triangle: a) Invent numbers in the circles of the triangle in the way that the sum in each side row is the same. What is then the row sum? b) Invent numbers in the circles of the triangle in the way that the row sum is -10.

Since the written exam seemed to be a very easy one – the first pupils were ready about 15 min. before the end of the lesson – I developed the following additional problem from the number triangle that I wrote on the blackboard:

> *Investigate, whether it is possible to use multiplication instead of addition in the number triangle task (task 6). Thus: Invent numbers in the circles of the triangle in the way that the product in each side numbers is the same. If there is a solution, could there be several solutions?*

More than half of the pupils (about 60%) solved the task (Fig. 3) totally correctly. About one quarter of the pupils got zero points. And the rest was able to solve the task partially.

The additional problem was solved by about a third of the pupils: Some pupils found the trivial solution (0, 0, 0), and some others the solution (-2, -6, -8), but there was only one pupil who had produced several solutions.

Conclusion

Thus from the number triangle that I have planned to be a "disposable problem", it was surprisingly developed a large problem field. When dealing with the problems (during the 2.–5. lessons) all pupils were eagerly involved. A couple of pupils were especially motivated in this kind of action that could be seen e.g. in their willingness to discuss on their solutions also during their break time. The bored comment presented in the 6. lesson was no way representative for the general opinion in the teaching group, since it did not receive any support, but spontaneous objections. But according to my view, it is better to stop before the saturation, in order not to loose the "additional motivation" that the problem gave. Later on I had dealt with other kinds of problem situations in the same teaching group.

It is worthwhile noticing that in the beginning of the problem, it was used mainly positive integers (although the problem was not zoned in this way). The idea was that thus the pupils would more easily "get acquainted" with the problem situation. It easily can be seen that when the row sum is smaller than 52, there would be at least one negative number in the solution. And thus we are practicing the very topic (the addition and subtraction of integers) that belongs to the core curriculum of grade 7. Furthermore it is to notice that we have no way discussed all possible problem situations in the case of the number triangle. For instance, the case of fractions would be a very interesting one, both in the additive number triangle and in the multiplicative number triangle but, perhaps, later on in grade 8.

According to my understanding, events of the 5. lesson were surprising, when only a few pupils were able to rise on the level of conceptual thinking, although they had a concrete model for checking their thinking. Therefore, it would be a good idea to continue with the number triangle in the beginning of grade 8, when the calculations with integers will be repeated, and look at, whether there is any development on the level of pupils' thinking.

After the school experiment I have used about twenty years regularly the Number Triangle problem in teacher education, both in pre-service and in-service education courses. The problem offered me a good structure on the implementation of open problems in school instruction. And the results with teachers were very similar as in school, only teacher students and teachers grasped the ideas more quickly.

Final notes

The teaching unity Number Triangle was invented and probed almost 30 years ago, but its main objective (development of mathematical thinking) is still valid. Today we are talking much on digitalization (e.g. in the new curriculum, cf. NBE 2014), but much of mathematics should be still done mentally, only calculations can be left to computers. Furthermore, the promotion of mathematical thinking is an essential part of curricula in all countries around the world. We are able to develop pupils' mathematical thinking only by giving them proper problems to ponder, and before all reserve enough time for them to find solutions. Doing mere calculation tasks in school mathematics is not enough.

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