MATHEMATICAL PROBLEM SOLVING – ON SOME CONSTRAINTS IN TEACHING AND RESEARCH

BERND ZIMMERMANN
Friedrich-Schiller-University of Jena
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"Some study theories of relativity,
some study relativity of theories."
(Unknown author)

Teaching of problem solving as well as research in problem solving are subjected to many different constraints and hidden assumptions. Many of the constraints for teaching hold as well for research in these areas. The detection and the analysis of such issues are carried out on the background of concrete elucidating examples ("proofs of existence"). We conclude with some theses, related to such constraints as "As well teachers as researchers should be much more aware on their respective belief-system" and "There should be more analysis and reflection about possible constraints and limits on own studies on problem-solving and metacognition." Some own reflections on possible constraints of this study are presented as well.

INTRODUCTION

In this contribution, I want to highlight some constraints in teaching and research of mathematical problem solving. Teachers as well as researchers should be much more aware of possible limits and constraints of their respective enterprise. Such awareness might help to improve the planning and effectuation of their teaching or research project as well as its appropriate evaluation. Evidence for the following findings is given by some concrete examples, some are taken from the literature, others from my personal experience in both areas. Thus, methods applied here to analyse methods of teaching and research are careful analysis and "proof of existence", which can be taken as appropriate methods for such kind of meta-study (cf. Schoenfeld 2007, 2008). Extremely useful for the generation of the theses presented here proved to be my personal style to gain experience ("learning-style") subjecting myself to strong contrasting environments (as between the former

Western and Eastern Germany; as between underprivileged, low achieving pupils and mathematically talented pupils). One can say that focussing on differences in underlying norms and values and awareness concerning such issues constitute the norms and values (the "theoretical background") guiding this study. Because of space-limitation, a more detailed analysis of norms and values in mathematics education cannot be given here. Such analysis was started in more detail several years ago in connection with an attempt to develop a metatheory of mathematics education (Zimmermann 1979, 1981, 1983). To some aspects of this point I will relate later.

CONSTRAINTS OF TEACHING MATHEMATICAL PROBLEM SOLVING

The enterprise of teaching mathematical problem solving is subjected to many constraints.

I want to demonstrate this thesis by an example, taken from the dissertation of Fritzlar 2004.

Fold-and-cut-problem:

1. Take a sheet of paper of A4-format. Fold the paper along the symmetry-axis to the long sides, and cut off the adjacent corners.
2. Do not unfold the paper!
3. How the paper will look like after unfolding? Could you find some structures?
4. Unfold the paper now.
5. Repeat the process by folding now the paper along the axis vertical to the first one and cut off the corners.
6. Do not unfold the paper before making conjectures about the structure of the paper after unfolding. Try to give arguments!
7. After this, check your assumptions by unfolding the paper.
8. Repeat the whole process as often as you want or can do.

As a part of his PhD-work, Fritzlar presented this problem[1] in this or similar

[1] This problem was invented by Professor Kießwetter to test mathematical giftedness of applicants for teacher studies who have no high-school examinations. Kießwetter is especially known for his fostering-project for mathematically talented pupils, which runs very successful now for more than 34 years at Hamburg University (main focus on mathematical creativity, cf. Wagner&Zimmermann 1986).

versions in some 50 classes of grade 4 or 5 in Germany and got a huge bulk of very interesting responses and ideas of pupils who worked on this problem individually or/and in collaborative way.

Some of them were stimulated by the following sequence:

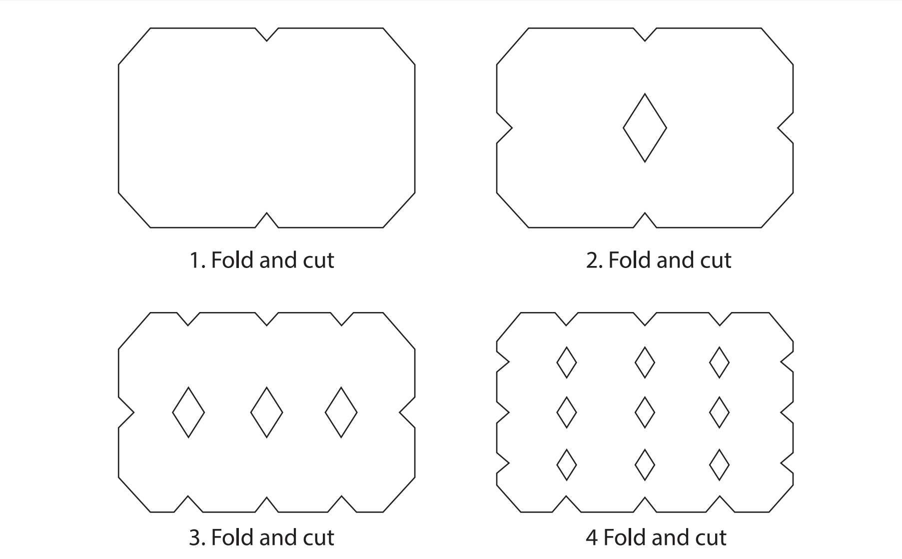

Figure 1: Sequence of first four fold-and-cut steps, cf. Fritzlar 2004, p. 366.

Some pupils focused on holes. For some of them the first hole did not only appear after the second fold-and-cut step. Because after discussing the question "What is a hole?", some pupils claimed that there are two "half-holes" and four "quarter holes" already after the first fold-and-cut step. Therefore, they got in this way already one total hole after the first fold-and-cut step! Some pupils concentrated on the "complement" of such "holes", the scraps of paper (see also Nolte 1995, p. 93). Altogether, the pupils created very different ways of counting.

Here are some more examples which were generated by placing emphasis on the folding-lines and the tiles detected by these:

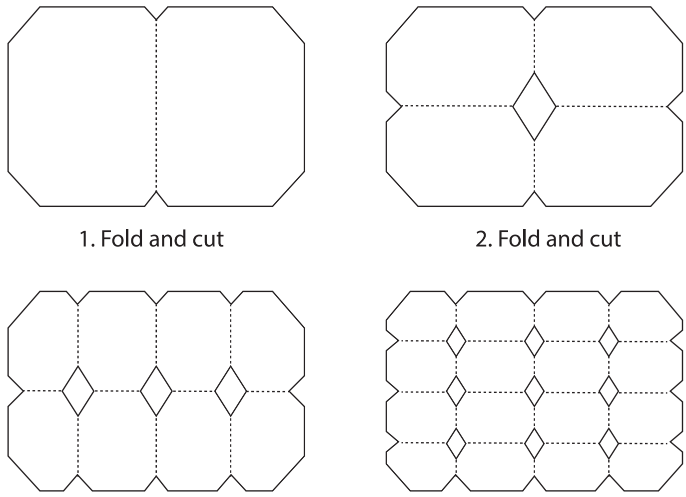

FIGURE 2: First four folds and cuts with folding lines.

In this way, pupils found a beginning sequence of 2; 4; 8 and 16 tiles and generalized to the assumption “2^n tiles” for n folds and cuts. Furthermore, some figured out, that the “normal” holes are located at the intersections of the folding lines.

This finding can lead to the statement that the number of such holes equals the number of folding lines in horizontal direction times the number of folding lines in vertical direction.

Concentrating on the “total” holes, pupils set up the following table:

Number of fold- cuts	Number of holes
1	0
2	1
3	3
4	9

TABLE 1: Relation number of fold-cuts and number of holes.

These results stimulated the assumption that the number of holes equals the sequence of powers of three, therefore 27 holes were expected for 5 folds and cuts.

But this conjecture could be rejected at once by the next fold-and-cut step:

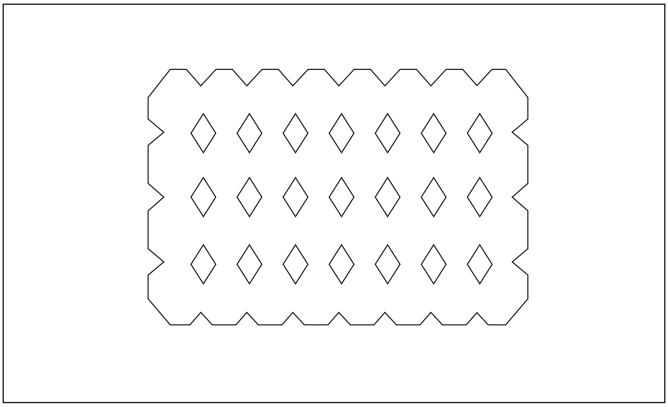

Figure 3: 5th fold-and-cut step, producing 21 holes.

Thus, the pupils experienced that conjectures can be rejected in a very easy way by further testing. This fact can challenge further thinking, which is another advantage of this problem.

But, there is a physical limit of testing by further folds and cuts (6 or seven steps is a maximum), depending on the stability of the scissors and on the properties of the paper.

Therefore, there comes another stimulus for the pupils to think ahead without further possibilities of concrete checking assumptions.

However, even this barrier could be broken by the pupils! Some of them got the idea that sheets of papers from classmates of the same fold-and-cut step could be put together to simulate the next fold-and-cut step (without additional folding and cutting)! In this way, several pupils could contribute in a very real sense to a possible solution of the problem!

Besides experience of success of many pupils there are some problematic experiences as well which demonstrate some constraints of teaching problem solving.

Some constraints of teaching

Most teachers of the experimental classes who were confronted with this folding- problem (the lessons were given by pre-service teachers or by the author of the study) were very reluctant to accept this problem as an appropriate one – in spite of successes of their pupils – because, as some of them said, such type of problems did not occur in their **curriculum**, so such problems seemed to be irrelevant for them.

Therefore, one might think that a curriculum with a clear focus on problem solving might do already to change this deficit (cf. the NCTM-standards, NCTM 2000). But there is evidence that this might be not enough.

First of all, in the preface of the corresponding math-curriculum of the aforementioned teachers (from the Federal State of Thuringia) problem-solving was already stressed as a leading idea for teaching. But it seems to be very probable for the author, that these teachers were used to focus only on the "coverage" of the following concrete content lists in the curriculum.

This experience might be related to the following one (from the US):

"There is a tendency of teachers to interpret new ideas and techniques through old mindsets" (Thompson 1992, p. 143, cf. also Zimmermann 1997). So, **teachers' beliefs** and their teaching styles they are used to as well as their reluctance to change them might be another important obstacle to teach more in a problem-oriented way (cf. Kilpatrick 1987, p. 300).

The **specific societal environment** has influence on teaching-reality at school, too. E. g., the role of authorities of the school board, individual independence, responsibilities and the degree of freedom to make own decision has been very different in Western and Eastern Germany before reunification. Until now these components might have still some influence on classroom-teaching.

Fritzlar 2004, pp. 3 – 8, reports about a lesson with the folding problem given by a preservice teacher for elementary schools to a class of grade four. Most pupils were very motivated and eager to fold and cut the paper and to work in "research-teams". Several "research-teams" presented their results very proudly on the blackboard.

Unfortunately, there was a severe lack of mathematical insight and results. One can conclude that different methods of teaching, motivation and enjoying concrete activities are to some extent necessary but not sufficient for a successful math-lesson.

Experience like this one help to demonstrate, that teaching and especially **teaching mathematical problem-solving** is a very **complex enterprise** (Fritzlar 2004, Zimmermann 2016). The teacher has to think about many important aspects as individual disposition of his or her pupils and their social background, curriculum, educational goals, motivation, appropriate intellectual challenge, time, next test etc.. Many of these issues are interacting, fuzzy or even hidden. So, teachers should be (or at least: become) sensitive for the complexity of teaching mathematical problem solving (Fritzlar 2004).

So, it makes great demand on teachers to teach mathematical problem-solving. This issue is closely related to the **educational and mathematical abilities of teachers**, which set further constraints to teaching of problem solving (cf. Zimmermann 2016).

Furthermore, when young pre-service teachers come back to school to teach mathematics after their university education, who are very well educated especially in teaching mathematical problem-solving, it should be possible that one could expect some change in the teaching culture at schools. But one must not neglect the **pressure of adaptation** from the experienced colleagues teaching already mathematics for a long time. This issue is related to the very well-known issue of "double-discontinuity" (Klein 1933, p. 1, cf. also Klein 1908/1939) in the career of students who want to become math-teachers. When going from school to university students can experience that they have to forget all about that kind of mathematics they learned at school. When going from university back to school to teach mathematics they are often told from old experienced colleagues that they have to forget all about that "theoretical nonsense" they learned about abstract mathematics and modern didactics. Finally, they have to teach very often mathematics once again in the well-known manner they learned already during their first time as pupils at school. So, university-education is frequently memorized as a nice intellectual experience without any influence on his or her teaching practice. At ICME 13 in Hamburg this issue was addressed by a special "topical survey" (Gueudet et al. 2016). But researchers talked mainly about the first problem (transition from school to university), but hardly about the second one (university to school). There was only some discussion about theoretical concepts as pedagogical knowledge (PK), content knowledge (CK), and pedagogical content knowledge (PCK) as well as about possible outcomes of university math-teacher education (TEDS-M, cf. Gueudet et al. 2016, pp. 12 – 14). It is not clear in which way such studies help to uncover to what extent beginning teachers are exposed to forces of adjustments, provoked, e. g., by social environment of the school, experienced teachers, parents and administration. So, the transition from university to school seems to be still a strongly neglected "white spot" for research. But more knowledge about this issue would be indispensable to learn more about the effectiveness of teacher education and modes of implementation, not only with respect to mathematical problem solving.

What is mathematics? is another important question, related to the foregoing issues.

"The definition of mathematics changes. Each generation and each thoughtful mathematician within a generation formulates a definition according to his lights." Davis & Hersh 1981, p. 8).

There are not only different opinions about the nature of mathematics between mathematicians (cf. Burton 2004, Hadamard 1996, Poincare 1900, Zimmermann 2016a) but there are also differences in the use and understanding of mathematics in different cultures and times (Zimmermann 1998). Furthermore, we have different opinions about mathematics (beliefs) between teachers in different social environments and countries (cf. Leder, Pehkonen & Törner 2002, Zimmermann 1991).

I present another example from my experience in mathematical gifted-education from different environments, which highlights the existence of different conceptions of mathematics, mathematics learning and teaching.

When starting my new job as mathematics educator at the University of Jena in Eastern Germany, I visited a summer camp for mathematically gifted pupils (age 15 – 18), which was conducted by very talented and motivated mathematics-students from my university. The students hold lectures about some university-themes from mathematics, the pupils listened in full concentration and made notes. After the lecture the pupils posed some good questions. In the evening, I set together with the students who hold the lectures and we talked about our experience in gifted education. I referred to my experience in the Hamburg-project (Wagner & Zimmermann 1986). To clarify the philosophy of this project, I talked about the way we worked on the well-known Tower-of-Hanoi-problem. I told them that, after finding (and proving) the "classical" solution for the minimal number of moves, our main focus is to let pupils modify the problem. For example: How many moves would be necessary, if there were not only three places (or pegs) which can be used to deposit the tower or some of its disks but four or more places (Kießwetter 1985; as far as I know, the general problem is still unsolved until today). The students said that they never heard about such approach, where creativity is of major concern. In spite of their obvious talent and motivation for mathematics, they were not used to such way of doing mathematics.

The way of understanding mathematics and its teaching is also closely related to the question: **What is a (good) mathematical problem**?

Once again, there are different answers to this question, very often there is made no clear difference between a task and a problem (cf. Kilpatrick 1987, Zimmermann 1997). Anyway, there should be a barrier which hampers the problem solver to find a solution at once, e. g. by applying a learned solution-scheme.

When selecting appropriate problems, one has to take into account the **dispositions of the pupils** who should solve them, too. The decision, which kind of problems and challenge might be appropriate will depend not only on the specific age and abilities of the pupils but might depend also strongly on their social background.

Video-tapes from the TIMSS-Video-study some twenty years ago, demonstrate, that there can be not only major differences between "typical" teaching styles in different countries but also with respect to the intellectual demand of mathematical problems for the same grade e.g. in Japan, Germany and the USA.

Furthermore, there is some criticism on the quality of mathematical problems, for example on some problems from the first PISA-study (Kießwetter 2002, see above).

Finally, the **way of assessment and grading tests** has a major influence on problem- oriented mathematics instruction. First of all, the multiple-choice-format is exposed to critical discussions already for a long time (cf. Hilton 1981, p. 79, Lax&Groat 1981, p. 85).

But also in case of more open-ended formats for grading there are several difficulties and deficiencies (Kießwetter 2002).

Some years ago, I learned from a colleague who gave a talk at my university about difficulties in evaluating PISA-test items from the domain of language. He reported that he checked the achievements of the evaluators of the pupils' solutions of the specific PISA-Items. He could state severe deficits of the achievements of the evaluators, too. Therefore, there might be even more reasons to be critical towards PISA-results.

In summary, there are many constraints which can influence and sometimes can hamper sound teaching of mathematical problem solving. We try to classify the constraints mentioned so far and add some social aspects:

Pupils:

- social background
- abilities,
- picture of mathematics.

Teachers:
- education,
- belief-system,
- abilities,
- awareness of complexity of teaching,
- pressure of adjustment,
- understanding and selection of "good" problems
- picture of mathematics.

School-administration:
- "the" curriculum and its perception by teachers,
- kind of tests and way of assessment.

The social environment of schooling,
- How support of gifted and underachieving pupils is balanced?
- financial support.

CONSTRAINTS IN DOING RESEARCH IN MATHEMATICAL PROBLEM SOLVING

First of all, all constraints mentioned above with respect to teaching of problem solving can have impact on research in this domain as well. Therefore, the following section can be shorter.

Additionally, more constraints are determined by the way research or scientific work is understood in some parts of the community of math-educators. During the last time, there had been a **renaissance of quantitative empirical research** in mathematics education, too (cf. Schoenfeld 2008). This impression can be supported by the increase of the number of international comparative studies as TIMSS and PISA.

It seems to me especially interesting, in which way people, being in charge of such studies, underline the importance of such studies. Schleicher (2005) stated: "Without data, you are just another person with an opinion". If "data" is understood broad enough so that it can mean, e. g., also case studies, concrete examples used as a proof of existence or as a counterexample for a general statement or, a correct mathematical proof of a theorem, there should be no problem. But if this term should be understood as a claim that quantitative empirical studies as TIMSS or PISA or the only "scientific" studies, we cannot join this opinion. Such quantitative empirical studies can be subjected also to limits and shortcomings as insufficient content-validity (cf. Kießwetter 2002) or poor competence of evaluators (see foregoing chapter). Such narrow picture of science is often combined with the assumption that only data gained by operationalized and measured variables determine scientific reasonable "facts". But fundamental criticism of operationalizability and measurability is well known, cf. Houts1977 and the already mentioned criticism about standardized tests of Hilton 1981 and Lax&Groat 1981).

There is a comprehensive criticism of Freudenthal on quantitative studies (Freudenthal 1991, pp. 147-156, see also the chapter "What is science?" in Freudenthal 1978). He claims that one should place much more emphasis on qualitative research and must never forget the most important question "**What is the use of it**?" (for the consumer and not for the producer of research, Freudenthal 1991, p. 149).

Such constraints in doing research by overemphasizing quantitative empirical studies (and ignoring limits of such approach, too) reminds to the well-known "**positivism- dispute**" from the sixties (Adorno et al. 1976). There was some influence of this discussion on the discussions about attempts to constitute mathematics education as a scientific discipline in the seventies of the last century as well (cf. Bigalke 1974, 1978, Zimmermann 1979, 1981, 1983, cf. also the contributions to the Topic area "Theory of Mathematics Education" since ICME 5, held in Adelaide 1984). At least since the eighties one can observe a broadening of the scope of methods, which should be selected very carefully according to the respective research goal. In this context, we refer to Schoenfeld 2008, p. 479:

"…research methods are best chosen when one has some idea of what it is one is looking for. A research method is a lens through which some set of phenomena is viewed. …. Moreover, to continue the metaphor, different lenses are appropriate for different purposes—the same individual may use one set of glasses for close-up work, one for regular distance, and complex devices such as telescopes for very long-distance work. So it is with methods: the phenomena we wish to "see" should affect our choice of method…".

So, pluralism of research methods which fit to respective goals seems to be today common property of the community of mathematics educators. But it still remains very often a lack of sensitivity for respective constraints.

Furthermore, the so called "**theoretical frameworks**" of many studies should be better called "scientific belief-system" of the respective researcher, which focus much more on that what in some scientific communities is in vogue than on that background, which is important for possible users (which can be very different in different areas and situations). Sometimes there seems to be more "background philosophy without sufficient foreground" (Freudenthal 1978, p. 32).

During the last years, there is evidence, that **sound mathematical content** in empirical studies seems to **vanish slowly** (Jahnke 2010). To say it in a more metaphoric way: The evaluation tail is wagging the content-dog (cf. Haapasalo & Zimmermann 2015)!

Furthermore, there is still a **lack of research of effective implementation-strategies**

for better problem-solving (Zimmermann 2016).

There is a lot of research on problem-solving, reflection and metacognition, but **less reflection about conditions and limits of such research**.

So, the most important question is: **What is good research**? Some very interesting possible answers are given by Sierpinska and Kilpatrick (1992).

Both are presenting and discussing a set of important criteria of good research in mathematics education. Especially important are: **Relevance and originality**.

Furthermore, one should become conscious about the fact that every research - not only on mathematical problem-solving or other domains of mathematics education - is bound to **values** (see Freudenthal 1978, p. 30, Bishop 1988, pp. 71, Haapasalo & Zimmermann 1981, 2015).

Values can be very different depending on the background of the very researcher and user of research-results. But, anyway, they have to be made conscious, reflected, and they have to be negotiated. This issue was mentioned already in the introduction. It had been analysed and discussed in more detail by combining different approaches to mathematics education with different approaches to philosophy of science e. g. in Zimmermann 1979, 1981.

We summarize as follows:

- Constraints which can have impact on teaching of mathematical problem solving can have also influence on research in mathematical problem-solving.

- The specific philosophy of science determines what should be taken as scientific findings.
- Possible constraints of outcomes of empirical studies (on mathematical problem solving) are e. g. quality and adequacy of mathematical problems and evaluators.
- Methods used in studies (on mathematical problem solving) have often strong influence on the kind of goals which can be followed and the kind of mathematical content which can be selected and not the other way around.
- Hidden values and assumptions can have also some influence on studies on problem solving.
- Such values and assumption are often incorporated in that what is called theoretical background, which should be called – as to my opinion - better called scientific belief-system of the respective researcher.

CONCLUSIONS

We conclude with some final theses:

- As well teachers as researchers should be much more aware on their respective belief-system.
- When (teaching or) conducting studies on mathematical problem the respective goals and included mathematical content should determine the respective methods (of teaching and) research and not the other way around. In this way one might get more appropriate answers to the question "What is the use of it (research)?
- There should be better strategies to implement research on problem solving into the teaching of problem solving.
- There should be more emphasis placed upon relevant and original research than on such research which concentrates more on applying statistical methods in a "rigor" way on shallow content.
- There should be more analysis and reflection about possible constraints and limits on own studies on problem-solving and metacognition.
- Own reflections on possible constraints and limits of this study:
- I could present only some possible constraints, about which I became aware by experiencing specific concrete examples. There may be many other similar experiences. So, the discussed list of constraints cannot be complete.

- This study was influenced by my specific interest in and work on the relation between mathematics education and philosophy of science.

Finally:
"We sing of it and prize it,
We tell and hear about it,
We write of it and read of it,
And then we do without it."[2]

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[2] G. W. Leibniz, New Essays on Human Understanding. Bk 1, Ch. ii, quoted from Sierpinska 1992, p. 35

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