MONITORING AND GUIDING PUPILS' PROBLEM SOLVING

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This paper presents a discussion of the problem-solving approaches of primary and secondary school pupils in relation to the following issues: developing strategies, communicating, and receiving guidance. Guiding is the role of the teacher who should be sensitive enough to support pupils' thinking, when necessary, but not direct it. A group of pupils (35 pupils between 10 and 19 years old) were given a geometrical problem that required them to define the number of parts created when a single plane was divided by straight lines. Each pupil tackled the problem individually, while prospective teachers from the Faculty of Education observed and guided them. After analysing the prospective teachers' research reports on guiding pupils through the problem we came to the following conclusions: all the pupils needed guiding in order to make progress in problem solving towards general rule, most of the pupils need to learn about heuristics more systematically, prospective teachers got better inside view on thinking process for problem solving of different age groups of pupils. From the success at problem solving point of view we observed the following: until presented with a problem that required a geometrical approach, the differences among the age groups in terms of successful problem solving were not that noteworthy, the difference among age groups was observed in examples of more complex problem solving where a shift towards an arithmetical approach was needed.

Key words: *problem solving, generalisation, guiding, primary school, secondary school, prospective teachers, teacher's role.*

INTRODUCTION

Problem solving in the mathematics classroom is by no means new idea. Beside the fact that there are many researchers dealing with different ideas connected to problem solving (e.g. Pólya,1945; Reid, 2002; Cañadas and Castro, 2007;

Radford, 2008; Mason, Burton, and Stacey, 2010; Schoenfeld, 1985 ect.) who are in favour of encouraging problem solving among students, it is also the fact that problem solving is not accepted by the teachers and by those who develop teaching materials in the way as other topics in mathematics are (e. g. mastering written algorithms, solving equations...). Short analysis of the mathematics curriculums in most countries of the world proves that the students should solve problems and reason mathematically. It is very broad aim, in most cases not presented with examples and it is on the teachers to find their way to realize this aim. Some teachers help themselves with different sources of problems; some do the problems in their classroom only for the purpose of some kind of research. Why? "Problem solving is time consuming, it is not possible to carry it out with the whole group of students and we do not know what the students actually learn", is the typical answer we get from our teachers. Even if they try to do some problem solving, they find it interesting but not enough to implement their teaching with it. Problems in the mathematics textbooks (there are not many) are in most cases planed for the gifted students and are very often solved at home or individually after finishing the 'obligatory' tasks. These are in fact also not problems which develop inductive reasoning; they are a little bit more challenging than the others. With our present research in the area of problem solving we do hope to show again that problem solving is beneficial for the students and for the teachers. Firstly, prospective teachers were involved in the role of the teacher's researchers and secondly, the problem presented is interesting and appropriate for the students of different age.

To be able to conceptualize pupils' problem solving and the effects of giving hints in the process of solving problems, we are going to review the literature related to inductive reasoning and to the teachers' role in teaching and learning mathematics.

INDUCTIVE REASONING

Of all mathematical processes generalization is considered one of the most important ones. For some researchers generalization is what mathematics is about (Maj-Tatsis &Tatsis, 2012). We all generalize in our everyday lives. We probably do it considerably more often than we realise (Cockburn, 2012). According to Vinner (2012) generalization is the driving engine of the concepts in all domains and statements about almost any subject. Upon reflecting on people's thought processes, we can realize that there is a tendency to generalize.

Dorfler (1991) understands generalizing as a social-cognitive process, which leads to something general, and whose product consequently refers to an actual or potential manifold in a certain way. We need to distinguish between two aspects of generalization: seeing the general in the particular, or seeing the particular in the general (Kruetski, 1976).

In the first case we can speak of *inductive reasoning,* which is a very prominent manner of scientific thinking, providing for mathematically valid truths on the basis of concrete cases. Pólya (1967) indicates that inductive reasoning is a method of discovering properties from phenomena and of finding regularities in a logical way. He refers to four steps of the inductive reasoning process: observation of particular cases, conjecture formulation, based on previous particular cases, generalization and conjecture verification with new particular cases. Reid (2002) describes the following stages: observation of a pattern, conjecturing (with a doubt) that this pattern applies generally, testing the conjecture, and the generalization of the conjecture. Cañadas and Castro (2007) consider seven stages of the inductive reasoning process: observation of particular cases, organization of particular cases, the search and prediction of patterns, conjecture formulation, conjecture validation, conjecture generalization, general conjectures justification. There are some commonalities among the mentioned classifications. All of them include observation of cases, conjecture formulation and generalisation, but they also differ in stages of inductive reasoning. Reid (2002) believes the process to complete with generalization, whereas Polya adds the stage of »conjecture verification«, as well as Cañadas and Castro (2007), who name the final stage the "general conjectures justification". In their opinions general conjecture is not enough to justify the generalization. It is necessary to give reasons that explain the conjecture with the intent to convince another person that the generalization is justified. Cañadas and Castro (2007) divided the Polya's stage of conjecture formulation into two stages: the search and prediction of patterns and conjecture formulation. The above stages can be thought of as levels from particular cases to the general case beyond the inductive reasoning process. Not all these levels are necessarily present; there are a lot of factors involved in their reaching.

The second aspect of generalization refers to *deductive reasoning*. This is a process of inferring conclusions from the known information (premises) based on formal logic rules, whereby the conclusions are necessarily derived from the given information, and there is no need to validate them by experiments (Ayalon &

Even, 2008). Although there are also other accepted forms of mathematical proving, a deductive proof is still considered as the preferred tool in the mathematics community for verifying mathematical statements and showing their universality (Hanna, 1990; Mariotti, 2006; Yackel & Hanna, 2003).

There are many different important issues when discussing inductive reasoning. We are only mentioning the classification of the generalisation situations by Krygowska (1979: in Ciosek 2011) where she distinguishes among the following: generalisation through induction, generalisation through generalising the reasoning, generalisation through unifying specific cases and generalisation through percieving recurrence. Type of generalisation is dependent also on the nature of the problems.

There are so called procedural and conceptual problems which refer to procedural and conceptual knowledge. The research in this area was prominent in the years between 1980 and 2000 (see e. g. Skemp, 1979; Hiebert, 1986; Gelman & Meck, 1986; Tall & Vinner, 1981; Gray & Tall, 2001; Sfard, 1994). More recently Haapsalo (2003) defined *conceptual knowledge* as knowledge of a skilful "drive" along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or a rule) given in various representation forms. *Procedural knowledge, on the other hand,* denotes dynamic and successful utilization of particular rules, algorithms or procedures within the relevant representation forms (Haapsalo, 2003). We therefore can understand procedural problems as those which require mere procedural knowledge for their solving; in this case a problem solver is more focused on procedures, rules and algorithms. On the other hand, the conceptual problems are those which require the solver to be familiar with the specific mathematical concepts. We are also proposing that there are no disjunctive categories of problems in this manner: however, one of them (procedural or conceptual knowledge) prevails over the other one at problem solving; some kind of relation between procedural and conceptual knowledge must be established.

Another important stage in the process of generalisation - so called creative moment or abductive generalisation - was proposed by Peirce (1958 in Rivera et al 2007) and it is widely used in the recent research.

Radford (2008) uses a term abductive reasoning. He defines the step of noticing a commonality and generalizing it to the rest of the terms of the sequence as an abductive reasoning. Inductive reasoning is in relation to abductive phase defined as a phase of testing and confirming the viability of an abduced form (Rivera et al, 2007). Radford (2008) distinguishes between algebraic pattern generalizations and arithmetic generalizations. 'Algebraic generalization refers to capability of grasping commonality noticed on some particulars, extending this commonality to all subsequent terms and being able to use commonality to provide a direct expression of any term of the sequence (deduction of schema or rule)' (p.84). If the step of forming a meaningful algebraic rule in generalisation is missing then we talk about arithmetic generalization. There is another type of situation to deal with when the abductions don't result from inferring a commonality among the particulars, but are mere guesses. In that case abductions lead to guessing the expression for general case. Even if a general rule is formed it is not based on algebraic thinking but on guessing. Radford (2001) calls this type of generalization naive induction. Becker and Rivera (2006) also report about students' difficulties in producing a meaningful rule. They usually employ trial – error and finite differences as strategies for developing recurrence relations with hardly any sense of what the coefficient in the linear pattern represent. In the just explained terms this is not an algebraic but a naive induction. Similar results were found also by our recent research in this field (Manfreda Kolar, Mastnak and Hodnik Čadež 2012).

It is of course very important how students deal with the problems: do they work individually, in groups, are they guided by the teacher or not. Some research has been done in this manner, e. g. Rott (2013) who found out that there is a strong correlation between (missing) process regulation and success (or failure) in the problem-solving attempts (the pupils worked on the problems without interruptions or hints from the observers). We are in this research among others interested in the role of the teacher if he is guiding the pupil through problem solving: what kind of hints are appropriate in the problem solving process and how do the problem solvers communicate about mathematical ideas.

TEACHER'S ROLE IN TEACHING PROBLEM SOLVING

In general there are two types of teaching approaches: traditional and non traditional which different researchers name for example as progressive, enquiry, participatory learning, student-centred learning (Brandes & Ginnis, 2001). Bennet (1976) stated that teaching problem solving requires student-centred learning with its typical characteristics: teachers have the role of a guide to educational experience, pupils participate, learning is based on discovery techniques, there is an accent on cooperative group work, creativity, process is valued and cognitive and affective domains are equally important. When we speak about student-centred learning the teacher and learner are having the interactive process where they operate in an interactive two-way process; the teacher acting as the sensitive facilitator and the learner retain overall ownership of the process (Brandes & Ginnis, 2001). This process must be learned or in other words teachers must move from established well-known ground to explore new teaching strategies. The teacher's role must be transformed from an instructor to a tutor, someone who guides students in the learning situation. This is rather a difficult task. Kuzle and Conradi (2016) for example reported about observing teaching practices. One of the areas of observations focused on problem solving, with two items: dealing with problematic statements or problems and students' use of problem solving strategies. They concluded that the area of problem solving was poor: when the problems were introduced they were primarily done by the teachers (Kuzle, Conradi, 2016). According to Kilpatrick (1985) there are five processes which play an important role in developing problem solving instruction: students solving the problems themselves, teaching heuristics, modelling problem solving situations, working in small groups, reflecting on the progress in problem solving. We must mention also Polya (1967) and his well known four phases model for problem solving: understanding the problem, devising a plan, carrying out the plan and looking back. We know that problem solving is not a linear process, it is rather dynamic, involving going back and forth, devising different plans, failing and trying again. The process of solving problems is also very subjective (each learner approaches to a given problem in his own way) therefore managing the classroom is very challenging. There are at least four factors contributing to success or failure in problem solving (Schoenfeld, 1985): resources (learner's conceptual and procedural knowledge), heuristics, metacognition (checking results, application of heuristics, monitoring the process), and beliefs (refer to learner's view about self). If we think of a teacher managing all of these factors we agree that this is a very complex task. Teachers' main role is to organise learning processes that enable pupils to master conceptual and procedural knowledge, to show different heuristics (looking for patterns, drawing figures, examining special cases, making tables...), to encourage pupils to monitor their problem solving process also by demonstrating the process while problems solving with the pupils frontally, and to make pupils believe that they can manage to solve problems by giving them appropriate challenges. Problem solving is therefore a 'topic' in mathematics which must be taught preferably in a student-centred learning setting. The fact is that successful problem solving will not happen by itself, it must be carefully planned in order to be successful for the teachers and the learners. We also claim that the teachers must first understand the processes which take place when pupils are involved in problem solving. In our opinion this can be done better if the teacher first works with the problem solver on an individual basis. Based on these assumptions we organised our research in such a way in order to help future teachers to better understand the pupils' problem solving processes, to experience their roles in the problem solving processes and to reflect about their guiding pupils through problem solving. We believe that this process might help future teachers to be motivated to organise problem solving situations in the classroom.

EMPIRICAL PART

Problem Definition and Methodology

The mathematics curriculum for primary schools in Slovenia includes a number of goals related to problem solving. For example, a specific goal is articulated in the section about arithmetic operations stating that pupils should be able to use arithmetic operations in problem solving. In the section of the curriculum entitled didactical recommendations, it is explained that problems must be understood as tasks where the solver does not know the strategy in advance, but has to develop a strategy in order to solve the problem. For the purpose of this paper we used the problem 'Lines and Parts in a Geometric Shape' that would give deeper insight into the pupils' problems-solving abilities. It is important to note that the pupils were not solving the problem on their own, but were observed and guided individually by prospective teachers whose goal was not to be overly suggestive, but rather to provide appropriate hints, challenges, and comments.

The empirical study was conducted using the descriptive non-experimental method of pedagogical research.

Research Questions

The aim of the study was to answer the following research questions:

1) What type of generalisation of a given problem do pupils of different age perform?
2) How do prospective teachers guide pupils through problem solving?

Sample Description

The study was conducted at the Faculty of Education, University of Ljubljana, Slovenia in 2015. It encompassed 12 research reports, written by students studying to become primary school teachers (2nd cycle degree). Each prospective teacher worked on the problem 'Lines and Parts in a Geometric Shape' individually with two to three pupils at different grade levels. Each student chose their own sample of pupils according to the following guidelines: one pupil from lower primary school (grades 4-6), one from upper primary school (grades 7-9), and one from secondary school. There were no other requirements for choosing the sample. The total number of pupils in the sample was 35 (14 from grades 4-6, 11 from grades 7-9, and 10 from secondary school). None of the pupils was low achiever or identified as gifted for mathematics. The research was taken at pupils' schools in an urban area.

Data Processing Procedure

Data processing procedure consisted of two phases:

First, the prospective teachers were given a problem 'Lines and Parts in a Geometric Shape' during the elective course 'Selected topics in didactics of mathematics'. They had time to solve the problem individually, then they discussed their ways of thinking in pairs and afterwards different strategies were presented and discussed. In this way they deepened their understanding of the problem. We have to mention that none of them was able to achieve algebraic generalisation on his own.

Second, the pupils were given a mathematical problem and the prospective teacher observed how they solved it and what stages of generalization they achieved. The prospective teachers then wrote a protocol for guiding pupils' problem solving. They also wrote reports that included the pupils' work, dialogues with the pupils, and hints given to the pupils in order to support their problem solving.

The following is the problem 'Lines and Parts in a Geometric Shape' that was given to the pupils:

> You have a rectangle. If you draw one straight line across the rectangle, you will have two parts. If you draw two straight lines across the rectangle, you will have three or four parts.

Your task is to get as many parts as you can when crossing the rectangle with three, four, five straight lines. What about if you have 15 lines or n lines?

We believe that this problem has the potential to contribute to the following aims: to reason mathematically and to communicate in mathematics, to discover new connections among different ideas in mathematics, to represent graphically different situations considering lines and shapes, to experience the change of mode of representation (geometric into arithmetic representation and vice versa), to identify mathematical patterns and to generalise.

RESULTS AND INTERPRETATION

1) *Pupils' performance of solving the problem*

First we present some general conclusions about the success of pupils' problem solving. There is almost no difference among grade levels for the first three cases of the problem (3, 4, and 5 lines). In the case of 15 lines, there is a significant difference, but it was skewed by the fact that there were many secondary school pupils who did not work on this level of the problem. The rate of success among 4-6 graders was 86 %, among 7-9 graders 73 %, and among secondary school pupils 70 %. The most significant difference in success rates by grade level is seen in the case with n lines. None of the pupils in grade 4 to 6 began to generalize this case, 9 % of 7-9 graders generalised to n lines, and 60 % of the oldest pupils in the sample. This was in part due to the decision of the prospective teacher who guided them to not present the case with n lines, because they predicted that pupils of that age and with limited knowledge of algebra would not be able to solve it.

We summarized the results of the pupils' problem solving also according to the stages of inductive reasoning by Cañadas and Castro (2007). Pupils' strategies for solving the problems with up to 5 lines correspond to the first stage of inductive reasoning (observation of particular cases). The 5-line case pushed pupils to advance to the second stage of inductive reasoning: organization of particular cases. The 15-line case already represents the next stage of inductive reasoning: the search for and prediction of patterns, and the making of conjectures.

We came to the following conclusions after further analysis of pupils' generalizations. Pupils who produced oral explanations of the observed rule are at the stage of narrative generalization (Radford, 2008). Most of them also solved the problem symbolically. This means that they also arrived at the stage of either arithmetic or algebraic generalization. If a pupil solved the 15-line problem correctly but was not able to find a rule for the n-th case, we considered the stage of generalization to be arithmetic one. Only pupils who managed to articulate a general rule were considered to achieve the last stage: algebraic generalization. One of the pupils who produced a general rule for the n-line problem by guessing is considered to have relied on naive induction as defined by Radford (2008).

Table 1. shows the distribution of the level of reasoning achieved by the pupils.

Abductive reasoning	Narrative generalization	Arithmetic generalization	Algebraic generalization	Naive induction
34	23	27	6	1

Table 1: Distribution of pupils according to achieved stage of generalization by Radford (2008)

2) Guiding pupils through problem solving by prospective teachers

As already mentioned pupils did not work on the problems alone, but were guided by the prospective teachers. It was our main concern to analyse the prospective teachers' protocols for guiding the pupils and to know how many pupils needed hints and what kind of hints were the most helpful. Table 2 shows how many pupils needed to be guided for the cases with 3 to 5 lines.

3 lines		4 lines		5 lines	
71 %	86 % grade 4-6	63 %	86 % grade 4-6	63 %	86 % grade 4-6
	82 % grade 7-9		45 % grade 7-9		55 % grade 7-9
	40 % sec.		50 % sec.		40 % sec.

Table 2: Number of hints given for 3 to 5 line problems.

In general, more than 50% of all pupils needed guidance for all three cases. It is also clear that the younger problem-solvers needed more hints, the middle group needing the most hints for 3 lines but significantly less for 4 and 5 lines. We assume that the hints given for the first case helped the grade 7-9 pupils to solve subsequent cases. Deeper analysis of the problem solving indicates that pupils on this grade level were able to transfer strategies obtained for the first case to more advanced cases. For example, when solving the problem for 4 and 5 lines, they continued using the previous picture, not starting from the beginning as many of the youngest pupils did. It is interesting to note that the oldest group also needed hints, though significantly fewer than the younger groups.

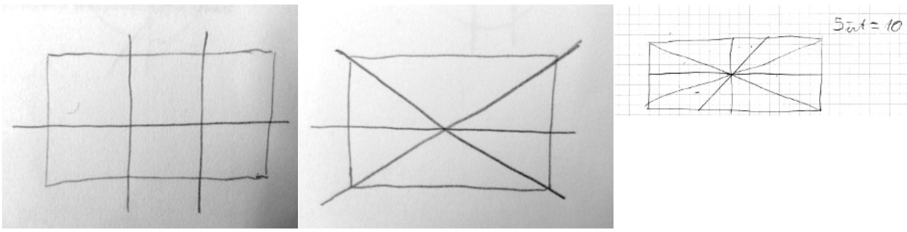
We were also interested in discovering what types of hints the prospective teachers used. Drawing on their protocols, we classified hints into nine main types of hints as presented below:

Type of hint	**3 lines**	**4 lines**	**5 lines**
Lines don't have to be vertical or horizontal.	8	1	
Lines don't have to intersect.	9	6	3
Count again.	1	2	3
There are more parts.	11	11	7
Help yourself by using the previous case.	2	6	1
Draw more precisely.		1	
Draw a bigger picture.		3	4
Make a table/organize data.			6
Each line should intersect all the previous lines			4

Table 3: Types of hints

We categorized the hints into two main groups: procedural hints (marked bold) and conceptual hints (marked italic). We can refer conceptual hints to Polya's first phase of problem solving protocol, namely understanding the problem, and procedural hints to the problem solving heuristics which are needed to solve mathematical problems. In general, the prospective teachers gave more conceptual hints (40) than procedural (20) hints, but the type of hint also depended on the number of lines. The most useful hint (there are more parts) is a general hint as it informs the problem-solver that the correct solution has not yet been achieved, but provides no information about the type of mistake that has been made or directions for solving the problem.

We can see that the number of conceptual hints decreases as the number of lines increases (3 lines: 19; 4 lines: 13, 5 lines: 8), whereas the number of procedural hints increases (3 lines: 1; 4 lines: 6, 5 lines: 13). When pupils began to solve the problems, they were mostly preoccupied with the position of the lines – they mainly used vertical, parallel, and horizontal lines – and the intersection of the lines – they insisted on drawing lines that intersected at one point (see Picture 1). The pupils seemed to need conceptual hints to move from their established perceptions of lines, the result of prototypes received in the teaching process, to less predictable positions.

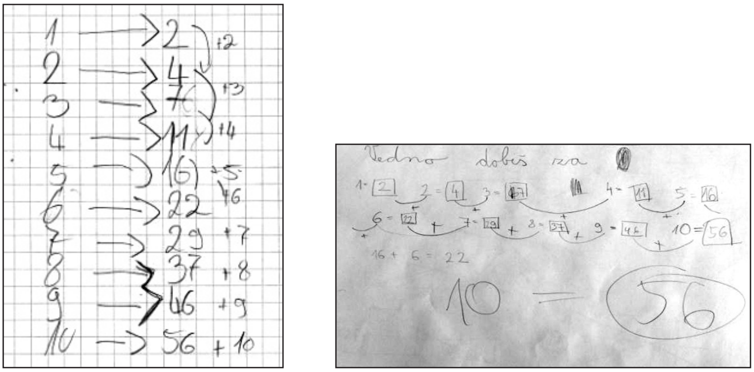


Picture 1: Typical mistakes with 3, 4 or 5 lines (horisontal lines, vertical lines and intersection of lines)

Once the pupils experienced this shift, they encountered another problem that was of a more procedural nature. Specifically, they had difficulties drawing more lines in addition to all the lines from the previous cases. These difficulties led the pupil, or the prospective teacher assisting the pupil, to find other ways to present the solution, for example, by organizing data in a table (see Picture 2).

To summarize our findings in relation to the teacher's role in the process of guiding pupils through problem solving we can conclude that the majority pupils were able to make progress in problem solving to higher stage only if they were given hints by the teachers (see Table 1). Without hints the pupils might stop solving the problem due to the misunderstanding of the problem's goal (for example focusing on parallel or intersecting lines).

On the basis of the presented hints in the process of guiding pupils' problem solving we can also conclude that all groups of pupils had similar conceptual problems with 3 lines and procedural problems with 5 lines. The stage of algebraic generalization was only achieved by the oldest pupils (secondary school pupils).



Picture 2: Examples of organising data

Table 4 lists heuristics used for solving the 5-line problem:

representation	5 lines
picture	19
table/organizing data	7
calculation	7
other	2

Table 4: Heuristics used for the 5-line problem

The 5-line case represents the point when a geometrical problem begins to change into an arithmetical problem. When solving the 15-line case, almost all pupils (34 of 35) recognized how the number of lines increased relative to all previous cases.

Let us show some examples of the protocols prepared by 2 prospective teachers, Jana and Tadeja. We translated their protocols in English and we tried to stick with their ways of expressing thoughts, hints, questions and conclusions. We chose these protocols of Jana and Tadeja because they very clearly reflected the prospective teachers' work with pupils and demonstrated our general findings about prospective teachers' guiding pupils' problem solving. First we present Jana's protocol with two pupils of different age and then Tadeja's protocol with the pupil of 19 years old. The examples of the interactive processes between the prospective teacher and a pupil will give a better understanding of the results presented above and interpretation of prospective teachers' guiding that follows.

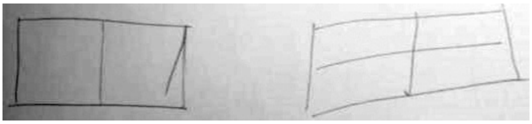
Jana's protocol

A girl A, 11 years old

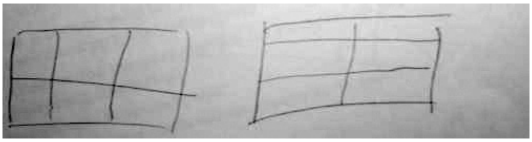
A girl first drew a line to get two parts and then drew another line and got 4 parts.

1 line **2 lines**

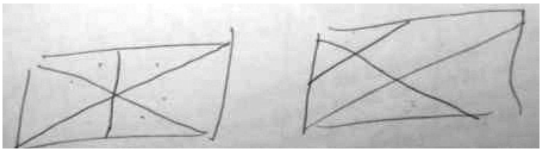
A girl drew three lines across a rectangle as presented in the pictures below. She used only horizontal and vertical lines and by using them she got 6 parts.

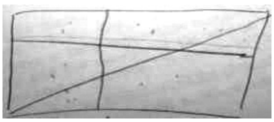


3 lines



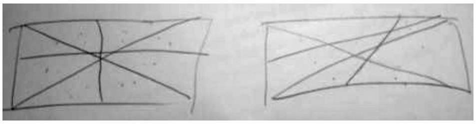
Hint: Lines do not need to be only horizontal and vertical. A girl drew the pictures presented below. In her third attempt she got 7 parts.

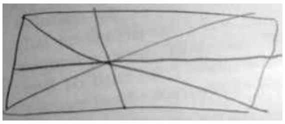




4 lines

When drawing 4 lines across a rectangle she got in all her three attempts 10 parts. When she counted parts in her third example she got only 8 of them due to her mistake in counting. She forgot to count the smallest parts.

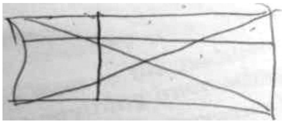




Hint: Count once more.

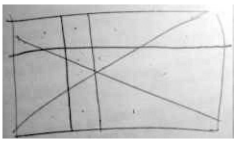
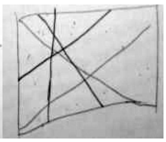
Hint: Can you find any other part.

She drew 4 lines across a rectangle once more and got 11 parts.

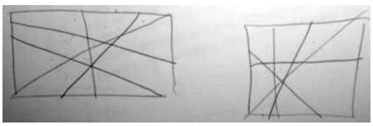


5 lines

She got some rectangles with 15 parts when she drew 5 lines across a rectangle. She forgot to count the smallest parts in her second example therefore she got 14 of them. In her last attempt she got 16 parts.

Hint: Count once more.



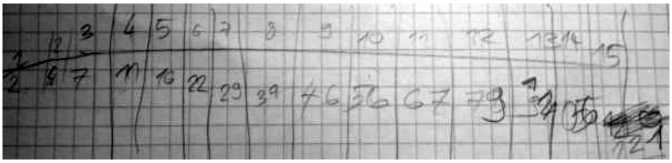
Additional question: can you find out how many parts are there if you draw 15 lines across a rectangle.

She wanted to make a picture and I asked her if this was appropriate. She responded that it was not appropriate because the picture would not be seen. I asked her if she knew any other way of getting the number of parts with 15 lines. After long time of her thinking I gave her a hint.

Hint: Make a table.

A girl found out that the number of parts increased by one in each case. She made a table for up to 15 lines. She made quite a lot mistakes in calculating.

Hint: Check if you calculated well.



How would you find out the number of parts with 25 lines?

She responded that by the same procedure. She would continue with the table.

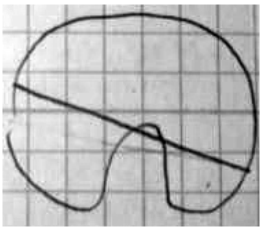
Then I encouraged her to find another, fastest way of getting the solution for 25 lines. She tied with the equation and she finished at this point.

<u>Additional question: What would happen if we have a circle instead of a rectangle?</u>

She explained that you get more parts with the circle because you can draw lines in a circle more closely.

I asked her to try out. She found out that we get the same number of parts in a circle as in a rectangle.

<u>Additional question: Is it possible to get a shape where you get more than two parts by crossing it with one line?</u>

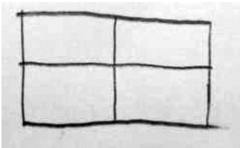


A girl B, 13 years old

You have a rectangle. If you draw one line across it you get two parts. How many part do you get if you draw two lines across it?

A girl got 4 parts with 2 lines.

2 lines



How many parts do you get if you draw 3 lines?

In her first attempt she got 6 parts.

Hint: Can you get any other part?

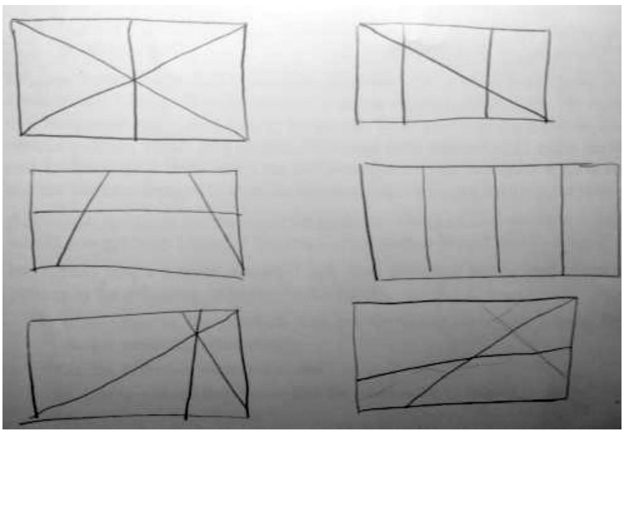
In her second and third attempts she again got 6 parts. In her fourth attempt she got only 4. I asked her why did she get only 4 parts.

Hint: Look at your previous examples.

A girl found out that in all previous examples the lines intersected, what was not the case in an example when she got 4 parts with 3 lines (picture below). She tried to draw another rectangle (5th in the picture below) where the lines again intersected. She found out that she again got 6 parts. I asked her what did the lines have in common. She responded that all lines intersected in one point.

Hint. Try to draw lines which do not intersect in one point.

She drew a rectangle (6th example in the picture below) and got 7 parts.



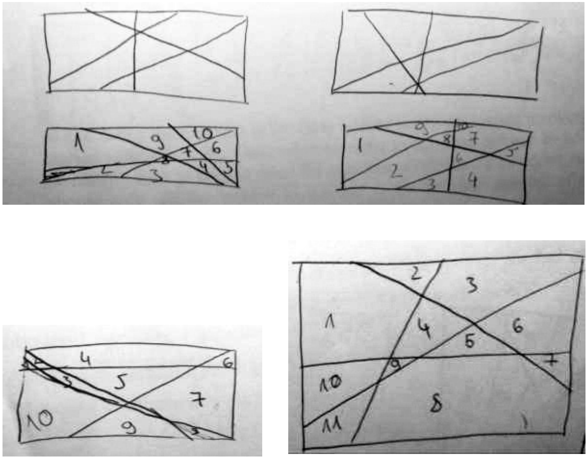
4 lines

Now, draw 4 lines across a rectangle. How many parts do you get?

In first attempt a girl got 8 parts, in 4 attempts she got 10 parts. Then she started to mark parts with numbers.

Hint: Try to draw a bigger rectangle.

She made a bigger rectangle and got 11 parts (6th example in the picture below).



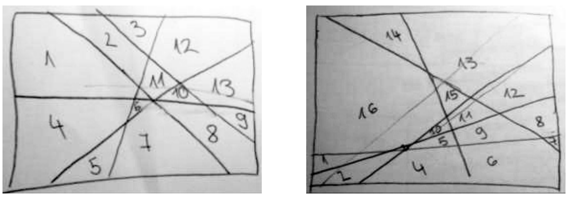
5 lines

How many parts do you get if you draw 5 lines across a rectangle?

Firstly, she got 11 parts.

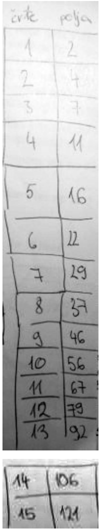
Hint: Try to find more parts.

A girls took long time to think how to arrange lines in a rectangle. She erased two drawn lines, drew them in a different way and got 16 parts.



<u>Additional question: Can you find out how many parts do you get with 15 lines?</u>

She made a table and calculated that you got 121 parts with 15 lines.



črte	polja
1	2
2	4
3	7
4	11
5	16
6	22
7	29
8	37
9	46
10	56
11	67
12	79
13	92
14	106
15	121

<u>Additional question: How would you find out the number of parts with 25 lines?</u>

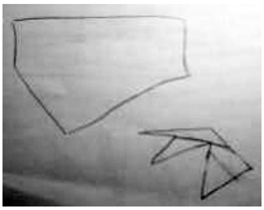
She responded that she would continue with a table.

<u>Additional question: What would happen if you have a circle instead of a rectangle?</u>

She responded that it would be the same; you draw the lines in the same way as in a rectangle.

<u>Additional question: Is there any shape where you can get more than two parts if you draw 1 line across it?</u>

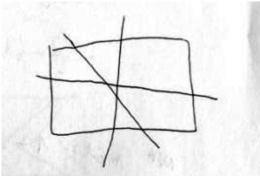
First she drew 5-sided shape (see the picture below). After long time of thinking I gave her a hint to try with different shape.



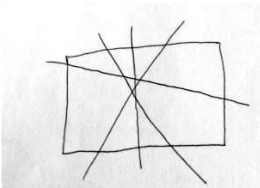
Tadeja's protocol

A boy, 19 years old

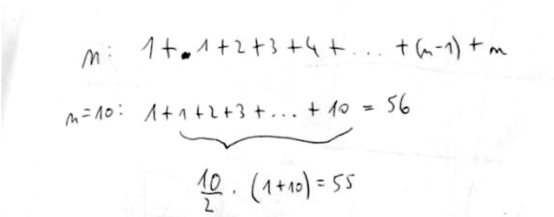
The pupil said he didn't want any help from the teacher, therefore he was solving a problem individually, I gave him only the basic instruction. A pupil got the solutions for 1, 2 and 3 lines.



He needed some more time for getting the number of parts with 4 lines but he managed to find the correct solution.

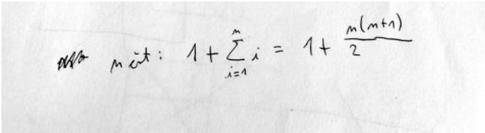


He found out the solution for 10 lines without using any picture or table, only by reasoning as presented on the picture bellow:



From pictures for 1, 2 or 3 lines he formed the following rule: each new line has to intersect all the previous ones and at the same none of three lines should intersect in the same point.

He wrote the formula for n-lines directly and transformed it into the formula for addition of n consecutive natural numbers



By analysing these protocols we can conclude that prospective teachers lead pupils quite directly, using the path they created themselves when solving the same problem[1]. They used very simple questions, not discussing a lot with the pupils. It seems that they wanted to bring them to the stage of solution as smoothly as possible. They did not encourage pupils of the first two age groups to generalise because they predicted that they wouldn't be able to perform it. Even with the oldest group of the pupils the prospective teachers didn't feel save to ask further questions. We can see that for example Tadeja finished at the point when a pupil of 19 years old produced a formula for n number of lines. From the protocols above we can see that the prospective teachers dealt with simple part of the problem quite easily, they were able to guide the pupils to the understanding of the problem, but further stages seemed to be more problematic: from the mathematics and from the guiding point of view.

CONCLUSION

In the conclusion we will respond to the following research questions:

How successful are pupils of different age in generalizing a given problem?

How do perspective teachers guide pupils through problem solving?

We found that pupils were able to solve the presented problems in many different ways, which had not been expected. In our opinion, all the strategies were interesting, innovative, and emerged from the pupils' prerequisite knowledge. This is especially true of the forms of generalization they were able to make. It became clear that we had to deal with different types of generalizations or reasoning as proposed by Radford (2008). Almost all pupils expressed abductive reasoning; it means they noticed that the number of parts is related to the number of lines by a certain rule. But only the oldest group of pupils was able to produce a general rule, i.e. they achieved the stage of algebraic generalisation.

[1] The prospective teachers were solving the same problem at the elective course on didactics of mathematics. The students who chose this course are motivated for learning mathematics and problem solving. We, as being also in the role of the teachers in this course, obeserved that the students had similar problems to those of pupils and were not able to generalise.

Throughout the project, we were aware that guiding pupils through the problem-solving process could be problematic, but we were also aware that, with thoughtful guidance, we can help pupils build their mathematical knowledge, self-esteem, and autonomy. According to Lev Vygotsky, this kind of collaboration is called 'the zone of proximal development' (Vygotski, 1986).

As can be seen from the protocols, the prospective teachers posed very similar hints or questions to the pupils. Only two refused any guidance, wanting to solve the problems by themselves. The prospective teachers didn't want to help with the solution itself, only to guide the pupils in pursuing their own problem-solving approach. Some situations did occur where pupils became stuck, seemingly unable to make any further progress. In such situations, it is up to the teacher or researcher to decide how to proceed. Our analysis of prospective teachers' protocols of guiding pupils's problem solving gave a better inside view of this process. We cannot say that this was an interactive two-way process as proposed by Brandes an Ginnis (2001) but it was mere one-way process in terms of pupils being the learners and prospective teachers being the leaders of the learning process. We cannot speak about learning of both groups. We can confirm that an interactive two-way process must be learned or in other words teachers must move from established well-known ground to explore new teaching strategies. We believe that prospective teachers must develop competences in the area of mathematics as well as in area of heuristics, reflecting on the progress in problem solving. They need to get confidence for solving problems and allow themselves to learn from the pupils as well. As already mentioned problem solving is not a linear process, it is rather dynamic, involving going back and forth, devising different plans, failing and trying again. Our research contributes also to yet another confirmation of Schoenfeld's (1985) four factors contributing to success in problem solving. If we think of teachers instead of pupils that this factors become the following: resources (teachers' conceptual and procedural knowledge), heuristics, metacognition (checking results, application of heuristics, monitoring the process), and beliefs (refer to teacher's view about self).

We are aware that the research conditions under which the prospective teachers worked were ideal (i.e. working individually with the pupils), but nevertheless we do hope that it motivated them to start thinking about problem solving in the mathematics classroom in a different way – one that is more favorable to the learning process.

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