COVALIDITY AND COSATISFIABILITY

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Two formue of an elementary formal language are *covalid* if whenever one of them is true under all valuations compatible (i. e. agreeable on the common basic symbols) with a given valuation of the other formula, this other formula is true under that given valuation. This definition corrects the definitiion in *Bell and Machover* p. 98. Our definition of cosatisfiability agrees with that in *Bell and Machover* pp. 97-98: Two formulae are *cosatisfiable* if whenever one of them is true under some valution, there exists a compatible valution under which the other formula is true too. It is shown that *neither does covalidity imply cosatisfiability nor does cosatisfiability imply covalidity.*

These semantic relations are studied in comparison with the semantic properties of *validity* and *satisfiability.* Any two formule which are both logically valid or both are contradictions, are also covalid and cosatisfiable; but there are covalid or cosatisfiable formulae which are neither valid nor contradictory.

Special cases are considered in which covalidity and cosatisfiability reduce to:

(i) *bipartitions* of the set of all formulae *into valid and non-valid formulae* and *into contradictory and non-conctradictory formulae,* respectively;

(ii) *logical equivalence relation;*

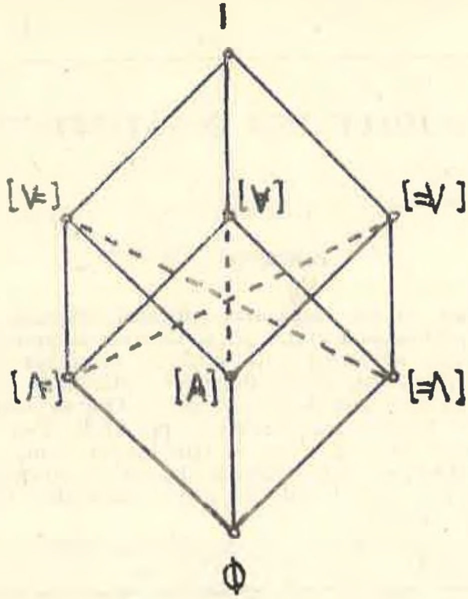
(iii) *validivalence and satisfivalence relations* (our terms) or (in a somewhat narrower sense) *V-form* (validity form) and *S-form* (satisfiability form) respectively (in terms of *Bell and Machover).*

We end the paper by briefly noting that what we have setforth constitutes a framework for the study of *Herbrand* and *Skolem forms,* and the problem of rearranging quantifiers in a given formula.

1. Validity and satisfiability

We shall be concerned with formulae of an *elementary formal language*, the alphabet of which consists of individual variables, function symbols, predicate symbols and proposition symbols, plus connectives and quantifiers (as logical constants), with the usual zero and first degree formation rules for terms and formulae. For any formula φ, φ^σ denotes the value of the mapping σ induced by an appropriate valuation, according to Tarski's *basic semantic definition*, i. e. $\varphi^\sigma \in \in \{\top, \bot\}$. (Cf. e. g. *Bell and Machover* p. 51).

Each formula belongs to one of the sets in the tripartition of the set of all formulae, |: the set of logically valid formulae (valid, for short), the set of contradictory formulae, and the set of formulae which are neither valid nor contradictory. For the sake of convenience, we shall denote these sets by [Λ⊨], [⫤Λ], [A] respectively, and their complements by [⫤V], [V⊭], [∀] respectively. Thus we have the following *field* (Boolean algebra) *of sets*. (This field is partially ordered by set-inclusion; complementation is indicated by dotted lines.)



There is an important semantic property of formulae, belonging to any given one of these sets, which we shall denote by prefixing formulae with the corresponding set symbol (omitting [,]). Thus:

$\bigwedge\!\models \varphi$ if $\varphi^\sigma = \top$ for *all* σ,
$\bigvee\!\models \varphi$ if $\varphi^\sigma = \top$ for *some* σ;
$\models\!\bigwedge \varphi$ if $\varphi^\sigma = \bot$ for *all* σ,
$\models\!\bigvee \varphi$ if $\varphi^\sigma = \bot$ for *some* σ;
$\mathsf{A}\, \varphi$ if $\varphi^{\sigma_1} = \top$ for some σ_1 *and* $\varphi^{\sigma_2} = \bot$ for some σ_2,
$\forall\, \varphi$ if $\varphi^{\sigma_1} = \bot$ for all σ_1 *or* $\varphi^{\sigma_2} = \top$ for all σ_2.

Since in logic truth has a definite priority before falsehood, the first two properties are especially designated; they define the standard notions of *validity* (commonly denoted by $\models \varphi$ (and *satisfiability* respectively. Obviously, the notions of contradiction and refutability could have been defined by the following schemes, which actually provide these notions their logical importance. (Double-line stands for iff.)

$$\frac{\models\!\bigwedge \varphi}{\bigwedge\!\models \neg \varphi}, \quad \frac{\models\!\bigvee \varphi}{\bigvee\!\models \neg \varphi}.$$

By the way we observe

$$\frac{\mathsf{A}\, \varphi}{\mathsf{A} \neg \varphi} \text{ and } \frac{\forall\, \varphi}{\forall \neg \varphi}.$$

2. Semicovalidity and semicosatisfiability

Now we turn to an investigation of binary *semantic relations* naturally related to the semantic properties just expounded. For any formula φ let $\mathscr{L}_\varphi$ denote the set of basic symbols in φ, (the »language« of φ), other than logical constants (i. e.

connectors and quantifiers). Hencefort we adopt the convention that σ evaluates φ and τ evaluates ψ. We shall say that σ and τ are *compatible* w. r. t. $\mathscr{L}_\varphi$ and $\mathscr{L}_\psi$ if they agree on $\mathscr{L}_\varphi \cap \mathscr{L}_\psi$ (cf. *Bell and Machover* p. 97). By definition, compatibility is a symmetric relation, but it is *not transitive*; for it may be the case that each of the three sets $\mathscr{L}_\varphi \cap \mathscr{L}_\psi$, $\mathscr{L}_\psi \cap \mathscr{L}_\chi$, $\mathscr{L}_\varphi \cap \mathscr{L}_\chi$ is not empty, and yet no two of them have non-empty intersections.

The definitions of the semantic relations announced are:

φ /⊨ ψ if whenever $\varphi^\sigma = \top$ for all σ compatible with τ,
also $\psi^\tau = \top$;

φ \⊨ ψ if whenever $\varphi^\sigma = \top$,
also $\psi^\tau = \top$ for some τ compatible with σ;

φ ⫤\ ψ if whenever $\varphi^\sigma = \bot$ for all σ compatible with τ,
also $\psi^\tau = \bot$,

φ ⫤/ ψ if whenever $\varphi^\sigma = \bot$,
also $\psi^\tau = \bot$ for some τ compatible with σ;

φ A ψ if whenever $\varphi^{\sigma_1} = \top$ and $\varphi^{\sigma_2} = \bot$,
also $\psi^{\tau_1} = \top$ for some τ_1 compatible with σ_1
and $\psi^{\tau_2} = \bot$ for some τ_2 compatible with σ_2;

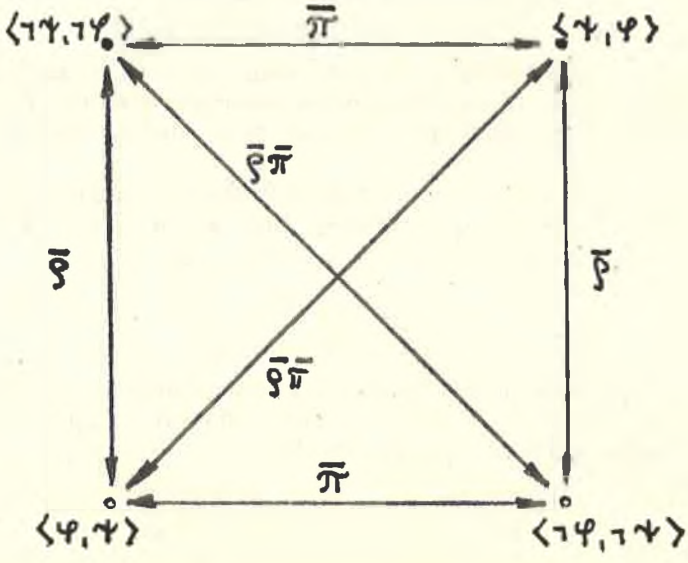
$\varphi \;\forall\; \psi$ if whenever $\varphi^{\sigma_1} = \top$ for all σ_1 compatible with τ_1
or $\varphi^{\sigma_2} = \bot$ for all σ_2 compatible with τ_2,
also $\psi^{\tau_1} = \top$ or $\psi^{\tau_2} = \bot$.

The first of these relations we call *semicovalidity*, the second *semicosatisfiability*.

Let $\langle \varrho\, \pi \rangle$ stand for any of the above relation signs, $\langle \bar{\varrho}\, \pi \rangle$ for the result of turning $\langle \varrho\, \pi \rangle$ upside-down, and $\langle \varrho\, \bar{\pi} \rangle$ for the result of turning $\langle \varrho\, \pi \rangle$ leftside-right. E. g. if $\langle \varrho\, \pi \rangle$ is /⊨, then $\langle \bar{\varrho}, \pi \rangle$ is \⊨, $\langle \varrho\, \bar{\pi} \rangle$ is ⫤\, and $\langle \bar{\varrho}\, \bar{\pi} \rangle$ is ⫤/. By means of straightforward checking, using definitions, we establish:

$$(1) \qquad \frac{\varphi \langle \varrho\, \pi \rangle \psi}{\neg\, \varphi \langle \varrho\, \bar{\pi} \rangle \neg\, \psi}, \quad \frac{\varphi \langle \varrho\, \pi \rangle \psi}{\psi \langle \bar{\varrho}\, \bar{\pi} \rangle \varphi}, \quad \frac{\varphi \langle \varrho\, \pi \rangle \psi}{\neg\, \psi \langle \bar{\varrho}\, \pi \rangle \neg\, \varphi}.$$

This is illustrated by the following »equivalence square«.



Furthermore, none of the relations $\wedge\!\!\models$, $\vee\!\!\models$, $\dashv\!\!\wedge$, $\dashv\!\!\vee$ implies (in general) any of the remaining three. This can be proved by means of counterexamples for

$\dfrac{\varphi \langle \varrho\ \pi \rangle \psi}{\varphi \langle \varrho'\ \pi' \rangle \psi}$, $\langle \varrho\ \pi \rangle$ and $\langle \varrho'\ \pi' \rangle$ distinct.

Since it is a consequence of (1) that each of $\vee\!\!\models$, $\dashv\!\!\wedge$, $\dashv\!\!\vee$ can be defined in terms of $\wedge\!\!\models$, we need only find counterexamples for

$$\frac{\varphi \wedge\!\!\models \psi}{\neg \varphi \wedge\!\!\models \neg \psi}, \quad \frac{\varphi \wedge\!\!\models \psi}{\psi \wedge\!\!\models \varphi}, \quad \frac{\varphi \wedge\!\!\models \psi}{\neg \psi \wedge\!\!\models \neg \varphi}.$$

Here they are (p and q are propositional variables): $p \wedge\!\!\models p \vee q$ holds, but $\neg p \wedge\!\!\models \neg p \wedge \neg q$ does not; $p \wedge q \wedge\!\!\models p \vee q$ holds, but $p \vee q \wedge\!\!\models p \wedge q$ does not; $p \vee q \wedge\!\!\models p$ holds, but $\neg p \wedge\!\!\models \neg p \wedge \neg q$ does not.

Similarly, $p \mathbb{A} p \wedge q$ holds, but $p \wedge q \mathbb{A} p$ does not, is a counterexample for

$$\frac{\varphi \mathbb{A} \psi}{\varphi \forall \psi}, \text{ which is equivalent to } \frac{\varphi \mathbb{A} \psi}{\psi \mathbb{A} \varphi}.$$

Also, $\mathbb{A}$ and $\forall$ are distinct from the previous four relations, for they satisfy

$$\frac{\varphi \mathbb{A} \psi}{\neg \varphi \mathbb{A} \neg \psi}, \quad \frac{\varphi \forall \psi}{\neg \varphi \forall \neg \psi}.$$

We could proceed with further examinations of the properties of these »one-way« relations, e. g. by analysing the sets $\{\varphi : \varphi \langle \varrho\ \pi \rangle \psi\}$ and $\{\psi : \varphi \langle \varrho\ \pi \rangle \psi\}$, but shall rather switch over to the logically more important »two-way« relations referred to in the title of the paper.

3. Covalidity and cosatisfiability

φ and ψ are *covalid*, $\varphi \wedge\!\!\models\!\!\!\dashv\!\!\wedge \psi$, if $\varphi \wedge\!\!\models \psi$ and $\psi \wedge\!\!\models \varphi$;
φ and ψ are *cosatisfiable*, $\varphi \vee\!\!\models\!\!\!\dashv\!\!\vee \psi$, if $\varphi \vee\!\!\models \psi$ and $\psi \vee\!\!\models \varphi$.

Our notion of cosatisfiability coincides with that in *Bell and Machover* pp. 97—98, while our notion of covalidity differs essentially from the one in *Bell and Machover* p. 98, and is probably what the authors of this book had in mind. What they said is:

»We shall say that α and α' are *co-valid* if whenever σ and σ' are a compatible $\mathscr{L}$-valuation and $\mathscr{L}'$-valuation respectively, then $\sigma \models \alpha$ iff $\sigma' \models \alpha'$«
The statement foollowing this definition, in our notation,

$$\frac{\varphi \wedge\!\!\models\!\!\!\dashv\!\!\wedge \psi}{\neg \varphi \vee\!\!\models\!\!\!\dashv\!\!\vee \neg \psi} \tag{2}$$

is true for our definition but not for theirs. Counterexample: $\neg p \vee\!\!\models\!\!\!\dashv\!\!\vee \neg p \wedge \neg q$ but for $p^\sigma = p^\tau = \perp$, $q^\tau = \top$; $\sigma \not\models p$ and yet $\tau \models p \vee q$, although σ and τ are compatible. To the contrary, by (1),

$$\frac{\varphi \wedge\!\!\models \psi}{\neg \psi \vee\!\!\models \neg \varphi} \text{ and } \frac{\psi \wedge\!\!\models \varphi}{\neg \varphi \vee\!\!\models \neg \psi}$$

prove (2).

Note that, since (again by (1))

$$\frac{\varphi \dashv\!\bigwedge \psi \;\&\; \psi \dashv\!\bigwedge \varphi}{\varphi \bigvee\!\!\models \psi \;\&\; \psi \bigvee\!\!\models \varphi} \quad \text{and} \quad \frac{\varphi \dashv\!\bigvee \psi \;\&\; \psi \dashv\!\bigvee \varphi}{\varphi \bigwedge\!\!\models \psi \;\&\; \psi \bigwedge\!\!\models \varphi}$$

we need not introduce new symbols for cocontradiction and corefutability.

The following schemes justify the names covalidity and cosatisfiability.

(3) $$\frac{\varphi \bigwedge\!\!\models\!\!\bigwedge \psi}{\bigwedge\!\!\models \varphi \text{ iff } \bigwedge\!\!\models \psi}, \quad \frac{\varphi \bigvee\!\!\models\!\!\bigvee \psi}{\bigvee\!\!\models \varphi \text{ iff } \bigvee\!\!\models \psi}.$$

It is in fact sufficient to prove second scheme, for it yields the first scheme. Indeed,

$$\frac{\varphi \bigwedge\!\!\models\!\!\bigwedge \psi}{\neg\varphi \bigvee\!\!\models\!\!\bigvee \neg\psi}$$ by (2),

$$\bigvee\!\!\models \neg\varphi \text{ iff } \bigvee\!\!\models \neg\psi$$ by (3) — 2nd scheme,

$$\text{non } \bigwedge\!\!\models \varphi \text{ iff non } \bigwedge\!\!\models \psi$$ by definitions,

$$\bigwedge\!\!\models \varphi \text{ iff } \bigwedge\!\!\models \psi$$ by contraposition.

Assume $\varphi \bigvee\!\!\models\!\!\bigvee \psi$. If $\bigvee\!\!\models \varphi$ i. e. $\varphi^\sigma = \top$ for some σ, then by the assumption also $\psi^\tau = \top$ for some τ (compatible with σ) i. e. $\bigvee\!\!\models \psi$. And *vice versa*. Thus (3) is proved.

However, the converse of (3) is false. Neither p nor $p \wedge q$ are valid; hence $\bigwedge\!\!\models p$ iff $\bigwedge\!\!\models p \wedge q$ holds, but $p \bigwedge\!\!\models\!\!\bigwedge p \wedge q$ does not. Both p and $p \vee q$ are satisfiable, hence $\bigvee\!\!\models p$ iff $\bigvee\!\!\models p \vee q$ holds, but $p \bigvee\!\!\models\!\!\bigvee p \vee q$ does not.

Hereon we can make some further conclusions. *Two satisfiable formulae need not be cosatisfiable*, as illustrated by the last example above. On the contrary, *every two valid formulae are covalid*. But, there are covalid formulae which are not valid, e. g. p and $p \vee q$. Notice that

(4) $$p \bigwedge\!\!\models\!\!\bigwedge p \vee q \quad \text{and} \quad p \bigvee\!\!\models\!\!\bigvee p \wedge q$$

do hold. Thus we have just shown that *neither does covalidity imply cosatisfiability, nor does cosatisfiability imply covalidity*: p and $p \vee q$ are covalid but not cosatisfiable; p and $p \wedge q$ are cosatisfiable but not covalid. To sum up: whenever $\varphi, \psi \in [\bigwedge\!\!\models]$ or $\varphi, \psi \in [\dashv\!\bigwedge]$, then $\varphi \bigwedge\!\!\models\!\!\bigwedge \psi$ and $\varphi \bigvee\!\!\models\!\!\bigvee \psi$; also, $\varphi \bigwedge\!\!\models\!\!\bigwedge \psi$ or $\varphi \bigvee\!\!\models\!\!\bigvee \psi$ for some, but not all, $\varphi, \psi \in [\underline{\Lambda}]$.

Relations (4), also indicate that for each formula there exist a covalid formula and a cosatisfiable, formula, neither of which is logically equivalent to it. (This is the theorem 10.5 of *Bell and Machover*, p. 99, but our proof is much simpler, for theirs utilises quantifiers). Indeed, for any formula φ and any propositional variable p not occurring in φ, we have $\varphi \bigwedge\!\!\models\!\!\bigwedge \varphi \vee p$ and $\varphi \bigvee\!\!\models\!\!\bigvee \varphi \wedge p$. Of course, $\varphi \bigwedge\!\!\models\!\!\bigwedge \varphi$ and $\varphi \bigvee\!\!\models\!\!\bigvee \varphi$.

At this point we adapt our remark concerning further examinations of properties of the elations $\bigwedge\!\!\models$, $\bigvee\!\!\models$ to the realtions $\bigwedge\!\!\models\!\!\bigwedge$, $\bigvee\!\!\models\!\!\bigvee$, thus proposing that the sets $\{\psi : \varphi \bigwedge\!\!\models\!\!\bigwedge \psi\}$, $\{\psi : \varphi \bigvee\!\!\models\!\!\bigvee \psi\}$ are worthy of studying. For example, by (3) it follows immediately that

$$\frac{\varphi \bigwedge\!\!\models\!\!\bigwedge \psi_1 \text{ and } \varphi \bigwedge\!\!\models\!\!\bigwedge \psi_2}{\bigwedge\!\!\models \psi_1 \text{ iff } \bigwedge\!\!\models \psi_2} \quad \text{and} \quad \frac{\varphi \bigvee\!\!\models\!\!\bigvee \psi_1, \text{ and } \varphi \bigvee\!\!\models\!\!\bigvee \psi_2}{\bigvee\!\!\models \psi_1 \text{ iff } \bigvee\!\!\models \psi_2}.$$

4. Validivalence and satisfivalence

So far we had in mind a general case that

$$\mathscr{L}_\varphi \cap \mathscr{L}_\psi \neq \Phi,$$

but also allowing the possibility that

$$\mathscr{L}_\varphi \cap \mathscr{L}_\psi = \Phi.$$

Besides, there is the opposite extreme case, namely

$$\mathscr{L}_\varphi = \mathscr{L}_\psi.$$

As will be seen, these two cases will give us what one could call standard situations. The most interesting, in our context, proves to be the case

$$\mathscr{L}_\varphi \subseteq \mathscr{L}_\psi \quad \text{or} \quad \mathscr{L}_\psi \subseteq \mathscr{L}_\varphi.$$

Bearing in mind that σ evaluates φ and τ evaluates ψ, we discuss each of these cases.

$$\underline{\mathscr{L}_\varphi \cap \mathscr{L}_\psi = \Phi.}$$

When this is the case, every two valuations σ and τ are compatible. Hence, by the appropriate definitions:

$$\begin{aligned} &\varphi \wedge\!\!\models \psi \text{ iff, } \wedge\!\!\models \varphi \text{ implies } \wedge\!\!\models \psi,\\ &\varphi \vee\!\!\models \psi \text{ iff, } \vee\!\!\models \varphi \text{ implies } \vee\!\!\models \psi;\\ &\varphi \;\; \wedge\!\!=\!\!\wedge \;\; \psi \text{ iff, } \wedge\!\!\models \varphi \text{ iff } \wedge\!\!\models \psi,\\ &\varphi \;\; \vee\!\!=\!\!\vee \;\; \psi \text{ iff, } \vee\!\!\models \varphi \text{ iff } \vee\!\!\models \psi. \end{aligned} \tag{5}$$

If we extend (5), as definition, to all formulae, $\wedge\!\!\models \varphi$ iff $\wedge\!\!\models \psi$ defines the bipartition of the set $|$ of all formulae into the set of valid formulae and its complement, while $\vee\!\!\models \varphi$ iff $\vee\!\!\models \psi$ defines the bipartition of $|$ into the set of contradictions and its complement.

$$\underline{\mathscr{L}_\varphi = \mathscr{L}_\psi.}$$

In this case σ and τ are compatible only if they are identical (of course, reative to $\mathscr{L}_\varphi$, or identically, to $\mathscr{L}_\psi$). Hence,

$$\begin{aligned} &\varphi \wedge\!\!\models \psi \text{ iff } \varphi \models \psi \text{ iff } \varphi \vee\!\!\models \psi,\\ &\varphi \;\; \wedge\!\!=\!\!\wedge \;\; \psi \text{ iff } \varphi \mid\!\models\!\mid \psi \text{ iff } \varphi \;\; \vee\!\!=\!\!\vee \;\; \psi. \end{aligned} \tag{6}$$

Here, as elsewhere, $\models$ denotes the standard relation of *logical implication*, and $\mid\!\models\!\mid$ denotes the standard relation of *logical equivalence*.

When extended to all formulae, (6) reads:

$$\varphi \mid\!\models\!\mid \psi \text{ iff } \varphi^\omega = \psi^\omega, \; \omega \text{ on } \mathscr{L}_\varphi \cup \mathscr{L}_\psi.$$

$$\underline{\mathscr{L}_\varphi \subseteq \mathscr{L}_\psi \text{ or } \mathscr{L}_\psi \subseteq \mathscr{L}_\varphi.}$$

In the former case compatibility means that σ is *the reduction* of τ (it is unique w. r. t. $\mathscr{L}_\varphi = \mathscr{L}_\varphi \cap \mathscr{L}_\psi$), while in the latter case compatibility means that σ is *an expansion* of τ (not unique w. r. t. $\mathscr{L}_\varphi = \mathscr{L}_\varphi \cup \mathscr{L}_\psi$). An application of the appropriate definitions to the situation $\mathscr{L}_\varphi \subseteq \mathscr{L}_\psi$ yields:

$$\varphi \mathrel{\wedge\!\!\models} \psi \text{ iff } \varphi \models \psi, \qquad \psi \mathrel{\vee\!\!\models} \varphi \text{ iff } \psi \models \varphi;$$

(7)

$$\varphi \mathrel{\wedge\!\!\models\!\!\wedge} \psi \text{ iff } \varphi \models \psi \text{ and } \psi \mathrel{\wedge\!\!\models} \varphi,$$

$$\varphi \mathrel{\vee\!\!\models\!\!\vee} \psi \text{ iff } \varphi \mathrel{\vee\!\!\models} \psi \text{ and } \psi \models \varphi.$$

Analogously, for the situation $\mathscr{L}_\psi \subseteq \mathscr{L}_\varphi$ we have:

$$\psi \mathrel{\wedge\!\!\models} \varphi \text{ iff } \psi \models \varphi, \qquad \varphi \mathrel{\vee\!\!\models} \psi \text{ iff } \varphi \models \psi;$$

(8)

$$\varphi \mathrel{\wedge\!\!\models\!\!\wedge} \psi \text{ iff } \varphi \mathrel{\wedge\!\!\models} \psi \text{ and } \psi \models \varphi,$$

$$\varphi \mathrel{\vee\!\!\models\!\!\vee} \psi \text{ iff } \varphi \models \psi \text{ and } \psi \mathrel{\vee\!\!\models} \varphi.$$

Of course, in (7) we understand that valuations τ are applied to both formulae bearing the relation $\models$, while in (8) valuations σ play this role.

In order to distinguish these cases, we introduce new symbols. Instead of $\varphi \mathrel{\wedge\!\!\models\!\!\wedge} \psi$ and $\varphi \mathrel{\vee\!\!\models\!\!\vee} \psi$ we write

(9)

$$\varphi \mathrel{\models\!\!\wedge} \psi \text{ and } \varphi \mathrel{\vee\!\!=\!\!|} \psi \text{ resp. in case (7);}$$

$$\varphi \mathrel{\wedge\!\!=\!\!|} \psi \text{ and } \varphi \mathrel{\models\!\!\vee} \psi \text{ resp. in case (8).}$$

In either case, if two formulae, whose sets of basic symbols are comparable, i. e. inclusively related, are covalid (cosatisfiable), then the formula with the smaller (larger) set of basic symbols logically implies the formula with the larger (smaller) set of basic symbols.

According to the terminology of *Bell and Machover* p. 94, for any formula φ, a formula ψ ($\mathscr{L}_\psi \supseteq \mathscr{L}_\varphi$) s. t. $\varphi \mathrel{\models\!\!\wedge} \psi$ is a *V-form (validity form) for* φ; a formula χ ($\mathscr{L}_\chi \supseteq \mathscr{L}_\varphi$) s. t. $\varphi \mathrel{\vee\!\!=\!\!|} \chi$ is an *S-form (satisfiability form) for* φ. (V-forms and S-forms are not uniquely determined by a given formula).

We shall extend (7) and (8) with symbolism (9) to all formulae, thus obtaining the definitions:

(10)

$$\varphi \mathrel{\models\!\!\wedge} \psi \text{ if } \varphi \models \psi \text{ and } \psi \mathrel{\wedge\!\!\models} \varphi,$$

$$\varphi \mathrel{\vee\!\!=\!\!|} \psi \text{ if } \varphi \mathrel{\vee\!\!\models} \psi \text{ and } \psi \models \varphi,$$

for any φ, ψ. In analogy with $\mathrel{|\!\!=\!\!|}$ being called *equivalence* relation, we shall call $\mathrel{\models\!\!\wedge}$ *validivalence* relation and $\mathrel{\vee\!\!=\!\!|}$ *satisfivalence* relation.

Trivially,

$$\frac{\begin{array}{c}\varphi \mathrel{\models\!\!\wedge} \psi \\ \varphi \mathrel{\vee\!\!=\!\!|} \psi\end{array}}{\psi \mathrel{|\!\!=\!\!|} \varphi}.$$

Equally trivially,

$$\frac{\begin{array}{c}\varphi \mathrel{\models\!\!\wedge} \psi \\ \varphi \mathrel{\wedge\!\!=\!\!|} \psi\end{array}}{\varphi \mathrel{|\!\!=\!\!|} \psi} \text{ and } \frac{\begin{array}{c}\varphi \mathrel{\vee\!\!=\!\!|} \psi \\ \varphi \mathrel{\models\!\!\vee} \psi\end{array}}{\varphi \mathrel{|\!\!=\!\!|} \psi}.$$

Naturally, all results on covalidity and cosatisfiability apply to validivalence and satisfivalence. In particular, $\varphi \mathrel{\models\!\dashv\!\wedge} \varphi \vee p$ and $\varphi \mathrel{\vee\!\models\!\dashv} \varphi \wedge p$. As an illustration that there are essentially different ways of obtaining V-forms and S-forms (by means of quantifiers instead of by means of connectors), we state

(11) $$\forall x\ \alpha \mathrel{\models\!\dashv\!\wedge} \alpha(x/c) \text{ and } \exists x\ \alpha \mathrel{\vee\!\models\!\dashv} \alpha(x/c),$$

where $\alpha(x/c)$ is the formula resulting from the uniform substitution of a new (not occuring in α) individual constant c for free occurrences of the individual variable x in α.

5. On Herbrand and Skolem forms

The investigations discussed in this paper constitute a framework for studying various kinds of validity forms and satisfiability forms for a given formula; especially the so called *functional* and *predicative* forms, and in particular *Herbrand* and *Skolem* forms. (In *Bell and Machover* pp. 94—97, Herbrand forms are also called Skolem forms).

In most textbooks the introduction of Herbrand or/and Skolem forms succeeds the introduction of the *prenex form* (cf. *Bell and Machover* p. 93), all this being in connection with rearranging the quantifiers in a given formula. A formula without quantifiers is usually called a matrix. For any (quantificational) formula we can construct its prenex form and its Herbrand or/and Skolem forms, (none of these forms is uniquely determined by the formula). In the prenex form quantifiers are separated from the matrix, and prefixed to the matrix. In Herbrand and Skolem forms existential and universal quantifiers in the prefix are separated, such that either all existential precede all universal (if any), or all universal precede all existential (if any) quantifiers. But, while the formula and its prenex form are equivalent, and this is usually made clear, the formula and its Herbrand or Skolem form are only validivalent or satisfivalent, and this is often not clearly indicated, (thus causing misunderstanding in the tsudent's mind). In our example (11) the formula $\alpha(x/c)$ is a Herbrand form for $\forall x\ \alpha$ and a Skolem form for $\exists x\, \alpha$, but neither $\alpha(x/c) \models \forall x\, \alpha$ nor $\exists x\, \alpha \models \alpha(x/c)$.

Ultimately, we call for advice on results and references concerning various rearrangements of quantifiers in a formula. We know that in a prenex normal form, though not unique, the status (i. e. $\forall$ or $\exists$) of each quantifier of a given formula is uniquely determined. How far can we get in separating $\forall$'s and $\exists$'s by means of forms logically equivalent to the given formula? Under which conditions are (certain levels of) separations or reordering of quantifiers possible? Are Herbrand and Skolem forms the optimal general results in this direction?

Reference

J. L. Bell and M. Machover, *A Course in Mathematical Logic*, North-Holland, Amsterdam, 1977.

Virgilio Muškardin: SUVALJANOST I SUZADOVOLJIVOST

Sažetak

Dvije formule elementarnog formalnog jezika su *suvaljane* ako istinitost bilo koje od njih za sve valuacije suglasne (za zajedničke osnovne simbole) s danom valuacijom druge formule povlači istinitost ove formule za danu valuaciju. Ova definicija ispravlja definiciju iz *Bell and Machover* str. 98. Naša definicija suzadovoljivosti slaže se s onom u *Bell and Machover* str. 97-98: Dvije formule su *suzadovoljive* ako istinitost bilo koje od njih za neku valuaciju povlači postojanje suglasne valuacije za koju je ona druga formula istinita. Pokazuje se da *niti suvaljanost ne implicira suzadovoljivost, niti suzadovoljivost ne implicira suvaljanost.*

Ovi semantički odnosi proučavaju se u usporedbi sa semantičkim svojstvima *valjanosti* i *zadovoljivosti.* Bilo koje dvije formule koje su obje logički valjane ili su obje protuslovne su također suvaljane i suzadovoljive, ali postoje suvaljane ili suzadovoljive formule koje nisu ni valjane ni protuslovne.

Razmatrani su posebni slučajevi, u kojima se suvaljanost i suzadovoljivost svode na:

(i) *bipartacije* skupa svih formula *na valjane i nevaljane* odnosno *na protuslovne i neprotuslovne* formule;

(ii) *relaciju logičke ekvivalencije;*

(iii) *relacije validivalencije i satisfivalencije* (naši nazivi) ili (u ponešto užem smislu) *V-forme* (forme valjanosti) i *S-forme* (forme zadovoljivosti) respektivno (u terminologiji *Bell and Machover).*

Primjedbom da je sve izloženo osnovni okvir za izučavanje *Herbrandovih* i *Skolemovih formi,* te nadovezanim problemom preuređivanja kvantifikatora u nekoj danoj formuli, završava članak.