ON THE NATURE OF MATHEMATICAL THEORIES

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This article deals with Putnam's philosophy of mathematics. Starting from Putnam's basic identification of the ontological attitudes of Platonism in modern mathematics, it quotes his example of Platonism (Russell's Principia). It propounds different possibilities of treatmant of mathematical knowledge.

(I)

Some Basic Philosophical Aspects of Platonism in Modern Mathematics

Trying to conceive, in the philosophy of mathematics, different kinds of realism, other then were proposed in Platonistic conceptions, Putnam isolated the foollwing basic *points* of Platonistic ontology:

(1) With mathematical theories, we describe mathematical objects: numbers, sets, functions, models. The ontology of mathematical true statements is the domain of such kind of objects. The difference between mathematics and the other sciences is in the abstract nature of the mathematical domain.

(2) Mathematical objects have a »modus« of absolutely real existence.[2] Mathematical theories show that, once accepted, mathematical truths are a actually existing invariable reality. They form an ideal mathematical »world«. Reality is somhow bifurcated into a physical reality and a reality of matematical objects.

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[1] We are dealing with Putnam's attitudes from the »first phase« of his philosophy. The period from 1964 to 1974.

Articles: »Trouth and Neccesity in Mathematics«, »The Thesis that Mathematics is Logic«, »Mathematic Without Foundations«, »What is Mathematical Truth«.

[2] In relation t othis Platonism Plato's philosophy of science is, in an ontological sense, to some extent, different. So, in relation to point (1): Plato could, fo course, think only of numbers and geometric figure that were known in his time. In relation to point (2): we can say that Plato would agree with this notion, but wiyh one supplement. Namely, according to Plato, the idea of number belongs to an ideal unchangeable world of ideas, but

(3) Mathematical knowledge is about something which is out of our minds.

(4) Mathematical knowledge »supports« the possibility of having *apriori* knowledge about the mathematical ideal reality.

For Putnam, the characteristic example of Platonism in modern mathematics is Russell's program given in his *Principia.* Let us briefly consult this example, from Putnam's point of wiew.

Russell's understanding of the modus of existence of propositional functions, according to Putnam, completely »covering« his *point* No (2). Making efforts to found mathematics on the principles of logical rules, Russell treats deductive quantification theroy in connection with the rules of propositional functions as a way to found the theory of mathematical entities. This way »logic« acquired a new content which, according to Russell, guaranteed possibility of reducing all mathematics to logic. Russell described the concept of »propositional function« as a function whose values are propositinal; propositional function is identical with geting universals in intension. Or rather, propositional functions are in a one-to-one correspondence in relation to predicates. In »light« of the theory of realism of universals, predicates are understand as real properties of mathematical objects. The ontology of mathematical true statements covers the domain composed of the sequence of abstract levels.[3] »Construction« of entities, which start from the zero level, include quantification theory and concept of identity. Basic entities are sets and individuals. Zero level demands logic of the first order, and the »construction« of complex numbers a logic of higher orders and predicates of the entities of zero level taken as arguments.[4] The concept

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physical reality is not completely bifurcated from this idea. The mathematical, ideal, so to say, »object picture« is the ontological basis, essence of physical reality. This is a supplement on Plato's understanding of the theory of knowledge as the recollection of the world of ideas (anamnesis). But, that means: having knowledge about the essence of the physical world. Possibility of apriori krowledge is extended to possibility of the apriori hnowlegde of the physical reality.

[3] Russell's zero level is used as the collection of the simplest individuals (which are not sets) plus collection of all propositional functions is understood as arguments ,plus ...

It is possible to show that Russell really gave to this kind of mathematic/logic the status of real existence with this quotation: »Every kind of knowledge is identification, maybe deceive, we must discover aritmetics as Colombus discovered West India, we don't create numbers more than he created Indians.« Russell B. Is Position in Space and Time Absolute or Relative, Mind, nb. X, 1901. p. 312.

[4] On the zero level, which is the frame for the tistinctive type of model, are individuals as the simplest »content« of mathematic/logic, which are defined as the result of the use of the concept of identity and rules of quantification theory. (I. e. statement »there is y such that x is one X ,and y is one X, and $x \neq y$ and for every r, if r i sone X than $r = y$ or $r = x$.«) On the first level entities are sets whose members are entities of the zero level,

»totality« as a property of really existing abstract entities Russell found in *Principia* in the frame of concepts: »totality of all predicates of integers«, »totality of all absolutely exsisting sets«. Such an antological status for the concept of the abstract mathematical entity is the result of, in Russell, the understanding of the concept of matematical model and especially standard mathematical model. Because of the unavoidable situation that the in *Principia* models are form ω-sequence Russell is forced to pressupose such properties as properties of undenumerable sets. He claims that such properies exist in the absolute sense of this concept; no matter that we are unable to produce evidence of these properties. Mathematical statements which refer to such predicates are, according to Russell, undubtely true. The truth value for these statements depends on the possibility of having access to the consisency of the pressuposed model.[5]

Putnam's criticism of Russell's Platonism:

The basic criticism is in the direction of the criticism of the totality of the real existing predicates functions: »We do not speak of a number two (in *Principia),* but of *the* number two. This is in agreement with the idea that there is some one definite mdoel which is presupposed in number theory, and that even the substitution of an *isomorphic* model would be a change of subject matter... Even if we took the numbers one, two, three,... as *primitive* (in direct violation of the Frege-Russell spirit), it would suffice to define 'A has *n* members' (where *n* is a *variable* over integers) to mean 'A can be put in one-to-one correspodence with the set of natural numbers less than *n*' (or alternatively, 'with the set of natural numbers from one through *n*). Then the equivalences P P discussed before would be forthcoming as theorems.«[6]

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etc. This extension of logic enabled Russell to define number as predicates of predicates (which have, as they understood as sets, cardinal number). This project is the result of Russell's efforts to avoid paradoxes. He established: a) Possibility of avoiding nonpredicative definitions which violates his »principle of the false circle«. b) The new concept of meaning and of sense in logic, because he proclaimed nonesense all statements which can't be fitted in to one of his types of existing models, c) The new concept of existence in the sense of the existence of only such abstract entities which are definable in his way. d) The concept of really existing abstract entities.

[5] As example, we can use Peano's arithmetic, understood as in Russell's sense of standardeness (in an absolute sense). Let us first mention that these five axioms of Peano's arithmetic can't cover the entire theory of natural numbers (there basic terms in this axioms do not include concepts of »set« and »pair«). Standardeness in this sense means a model in which every element is zero or successor of zero or successor of successor of zero. Properties; »be successor« and »be successor of successor« and... are absolute predicate functions, which, according to Russell imply existence of standard model for Peano's arithmetic in *Principia.* This means that it is possible to get proof of this model in »one of his types of entities« in *Principia.*

[6] Putnam H. Mathematics, Matter and Method, Cambridge Univ. Press, Cambridge 1975. p. 38.

Putnam tried to reinterpret Russell's attitudes from his earlier work (which Putnam call if-thenism) in such a way that it is possible to understand the nature of the concept of number exusively from empirical knowledge. This attitude of Russell is derived from the concept of identity and deductive rules of logic: »In order to solve this problem, let us abbreviate the statement 'the set of planets belongs to the number nine' as P_1, and the statement there is an x and there is an y and ... such that x is a planet and y is a planet and ... and x / y and ... such that for every z if z is a planet then $z = x$ or ..., 'which expresses 'the number of planets is nine' in purely first order way, as P*. The equivalence, P = P*, is a teorem of *Principia,* and hence holds in all models. *Thus, if we assume* ***Principia*** has a model, it does not matter whether we assert P or P*. Otherwise, as we wish to say without commiting ourselves to sets, models, etc.«[7]

Putnam criticised Russell's later project, expresed in *Principia,* which includes the concept of set as Platonism using the same criticism of Russell's basic notion of propositional function: »A standard model is here defined as above: one in which each element bears a finite power of the successor relation to 'zero' where the meaning of 'finite' may vary with the model selected for the set theory.[8] Putnam showed that the concept of »finite« really depends on the »kind« of selected model, on an empirical example: »And we can *fix* the notion of 'standard' model by taking this model to be *the* model. This is in effect what Kant did, but it is erroneus for just the reason that Kant' s views on geometry are erroneus: because the cosmological properties of time in the large are not more *a priori* than those of space in the large«.[9] Russell's

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[7] ibid. p. 31.

[8] ibid. p. 23. In note from 1974. Putnam no longer agrees with the »if — thenism« and argues that it is impossible to be an »if — thenist« in mathematics and an realist in physics. There are many possible reasons for this change in Putnam's attitude. Let us state some of them: i) »If-thenism« set up a very high degree of relativisation of the sense of mathematical assumptions which »sound« antirealistical in the understanding of the ontological status of mathematical statements. ii) Putnam is »moving« to his alternative possition: postulation of mathematical objects (even in such relative sense as in »if-thenism«) leads to Platonism or Antirealism. iii) »If-thenism« keeps the apriorism of the mathematical *episteme* and this does not allow the identification of mathematic and natural (aposteriori) sciences.

[9] ibid. p. 25. In notes from 1974. Putnam no longer agrees with that »concept of standardness has been undeefinirable, except in relitive sense«. This change of attitude »moves« in the direction of the definiability of the concept of standartness in the sense of modality (modal concept of possibility). This change is connected with the change in Putnam's understanding of the possibility of defining the concept of set; in the sense of taking the definition of the concept of set as clear in relation to the concept of mathematical possibility; which is »indispensable« and »irreducible«. In combination with Putnam's attitude of mathematic as aposterioral science, this claim shows that Putnam's »mathematical possibility« must be an empiric possibility.

concept of the »totality of sets« Putnam criticiesd in connection with the concept of undeterminateness: »Namely, there is a theorem of *Principia* which *says* that there are non-denumerably many sets of integers. Hence there must be non-denumerably many sets of integers (in any model). But this contradicts the Skolem-Löwenheim theorem, which says that *Principia* has a denumerable model (assuming *Principia* is consistent!)«[10]

(II)

Putnam's Proposal for Mathematics

Let us briefly look at Putnam's proposal for mathematics without postulating an abstract »objects« with one of his examples.[11] This example is in the function of his criticism of these philosophies which treat mathematics as a science with a special domain of act; respectively: »... the idea that 'ontology' (i. e. the domain of the bound variables) in mathematically true statements is a domain of sets or numbers or functions or other 'mathematical objects', and (moreover) that *this* is what distinguishes mathematics from other sciences is a widespread one.«[12] And: »This idea lives in constant tension with the other idea, familiar since Frege and Russell, that there is no sharp separation to be made between *logic* and mathematics. Yet, logic, has no 'ontology'!«[13] And from this follow: »It is precisely the chief characteristic of the principles and inference rules of logic that *any* domain of objects may be selected, and that any expressions may be instantiated for the predicate letters and sentential letters that they contain.«[14] It is not, accor-

[10] ibid. p. 15.

[11] Formalism is the kind of mathematical program which doesn't distinguish the special sense of mathematical objects. Putnam, however, doesn't agree with this program because he think that uninterpreted mathematical signs can't be treated, in any way, as realistic philosophy. Hilbert, also, identifies possible mathematical »objects« with signs. He understood mathematical activity as indepedent of anything but our rules of thinking, Quot.: »... the objects of number theory are for me ... signs themeselves, whose form may be generally and reliably identified by us indepedently of the place, time ,and the special conditions of the production of the signs ...« Hilbert D. Gesammelte Abhandlungen, Vol. 3. Springer, Berlin 1935.

[12] Putnam H. Mathematics, Matter and Method, Cambridge Univ. Press, Cambridge 1975. p. 1, 2.

[13] ibid. p. 2. It is evident that Ptunam, here, aligns mathematic very closely to logic (»It is not possible to distinguish mathematic from logic.«). But, he did not agree with the possibility of reducing mathematic theory on logic. Possible reasons are: i) logical reductionism didn't succeeded in the past, ii) this reduction, according to Putnam, invokes Platonism or Antirealism. The basic reason for this »drawing close« is, I think, in the fact that Putnam identifies logic as nonontological ,and, at the same time ,he is sharply against the foundational program in mathematics (nonontological attitude too).

ding Putnam, very difficult to make this confirm with example, because: »In point of fact, it is not difficult to find mathematically true statements which quantify only over material objects, or over sensations, or over days of the week, or over whatever 'objects' you like: mathematically true statements about Turing machines, about Turing machines, about inscriptions, about maps, etc.«[14]

Example: Let T be a physically realized Turing machine, and let P_1, P_2, ... P_n be predicates in ordinary language which describe its states. To is completely characterized by finite set of such instructions (E. g. If x P_2 (T) and T is scanning the letter »y«, T will erase the »y« print »z« in its stead, shift one square left on the tape and then adjust itself so that P6 (T) in »nominalistic« language. Call that instructions I_1, I_2 ... I_k. As long as I_1 and I_2 and ... and I_k, then T does not halt. Symbols I, II, III, IIII, ... designate the numbers one, two, three ... (i. e. the name of the number n is a string of n »I' s«). The sum of two numbers can be obtained by merely concatenating the numerals: nm is always the sum of n and m. The *meaning* of $x = y^+$ is: »x equals y cubed«.

nx *mean* »x is a number«.
! (shriek) indicate absurdity.

System E. S.

Alphabet: I, ., =, +, !, N

Axioms: 1. NI

2. $Nx \rightarrow NxI$
3. $Nx \rightarrow x = x$
4. $Nx \rightarrow x . I = x$
5. $x . y = z \rightarrow x . yI = zx$
6. $x . x = y,\ x . y = z\ z = x^+$
7. $z_1 = x_1^+,\ z_2 = x_2^+,\ z_3 = x_3^+,\ z_1 = z_2,\ z_3 \rightarrow\ !$

It is easily seen that! is a theorem of E. S. if and only if some cube is the sum of two cubes. Fermat proved that this is impossible.

So, the following statement (we can call it statement*) is true: If X is any finite suquence of inscriptions in the alphabet I, ., =, +, !, N, and each member of X is either an inscription of NI or of a substitution instance of one of the remaining above axioms, or comes from two preceding terms in the sequence by Detachment, then X does not contain!.

Possible objection to Putnam's attitude (that this example shows that mathematical true statements quantify only over physical objects)

[14] ibid. p. 2.
[15] ibid. p. 2.

is: even if some mathematical true statements quantify only over physical objects, still the proofs of these statements would refer at least to numbers, and hence to »mathematical objects«.

Putnam's response would be: The premise of this objection is false, because: if somebody wants to prove statement (which we call statement*) he needs the principle of Mathematical Induction. But, this can be stated directly for finite inscriptions, and it can be perceived to be evidently true when so stated. It is not that one must state the principle first for numbers and derive the principle for inscriptions via goedel numbering. This would assume that every inscription posseses a goedel number, which cannot be proved without assuming the principle for inscriptions. This is the principle stated as a rule of proof: If I, ., =, +, !, N are all P, and if for every x, if P(x) then P(xI), P(x .) . . ., P(xN), then, for every x, P(x).

There is no doubt that Putnam correctly shows that proofs of such statements as the stetament* is, (wich quantify only over physical objects as true mathematical statements), need only the principle of Mathematical Induction, and do not need mathematical objects. In other words : statement * as true mathematical statement refers only to physical objects (inscriptions).

But, what is the significance (meaning) of the axioms of E. S. which enable such statements as statements* is?

What will happen if we omit the concept N (means: number) from the axioms of E. S. (i. e. from axiom 1. — which is: »I is number«, or from axiom 4. which is: »x is number which imply that x is equal with x«, etc.)?

Or, what will happen if we omit this concept N from the premises of the rule of proof (i. e. from the premise »P(N)« — which means: »if N are all P«, etc.)?

The answers are perhaps these:

In that cases we will have the finite sequence of physical states (or physical objects) and »inscriptions« for objects (i. e. I, x → xI, x → x, I = x, . . . $z_1 = z_2 z_3 \rightarrow$!). But these are not even well formed formulas (i. e. x xI is either not a wff — »x« being a term and not a propositional variable, or false). We conclude: It is necessary to interpret »x« as number if we to treat System E. S. as a part of mathematics. This example was supposed to be a example of mathematics which quantifies only over physical objects. But, it does not follow that mathematics can function without its proper objects; its »peculiar« elements. There would be no mathematics at all! Because: *these elements belong to the »essence« of mathematics.*

So, the situation is as follow:

We can allow that some mathematically possible true statements quantify only overe physical statements, or objects, but; — this doesn't entail at all:

1. that it is possible to identify mathematics with science of physical objects and states,

2. that it is possible to separate mathematics from its »peculiar« elements. *They are an indispensable part of it.*

(III)

Concluding remarks

So far we have shown that there is no possibility of mathematics without mathematical entities.

What could these entities possibly be?

One answer is Platonistic. It seems natural and unavoidable, but I would like not to be constrained to accept it.

There is another, more exotic possibility — to treat mathematical knowledge as knowledge about some structures in our mind. Certainly, something in our brain organisation is responsable for our capacity to create mathematical theories. It is possible to understand mathematical theories as a body of knowledge which is an indirect way »about« those structures?

(What do we simulate with computers? — Really not something which is out of our mind.)

If this line of reasoning is correct, we would have a realism based on a physicalistic reduction of »mathematical« mental processes; to physicalistic reduction (physicalism) of »mathematical« mental processes to physiological processes. Certainly, such possibilities are distant and such assumptions far-fetched. Much more research is needed before we have a plausible answer.

Mirko Jakić: O PRIRODI MATEMATIČKIH TEORIJA

Sažetak

Ovaj članak se bavi Putnamovom filozofijom matematike. Pošavši od osnovnog Putnamovog identificiranja ontoloških stavova platonizma u modernoj matematici, naveden je njegov primjer platonizma (Russelova *Principia*). Njegov stav da postuliranje matematičkih objekata vodi u platonizam ili matematički antirealizam izložen je kritici.